Excitation of Accelerating Wakefields in Inhomogeneous Plasmas

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Abstract—The excitation of the wakefields in an inhomogeneous plasma by a short laser pulse is investigated theoretically. A general equation for the wake excitation in transversely nonuniform plasma is derived. This equation is applied to the stepfunction density profile model of hollow channel laser wakefield accelerator [1], [2]. A more realistic model, in which the transition between the evacuated channel and the homogeneous surrounding plasma occurs over a finite radial extent, is then analyzed. It is shown that the excited channel mode can interact resonantly with the plasma electrons inside the channel wall, leading to secular growth of the electric field. This eventually results in wavebreaking and the dissipation of the accelerating mode. We introduce an effective quality factor $Q$ for the hollow channel laser wakefield geometry. This resonance limits the number of electron bunches that can be accelerated in the wake of single laser pulse.

I. INTRODUCTION

ONE application of short-pulse laser-plasma interactions that has drawn considerable theoretical [3]–[8] and experimental [9] attention is the use of plasma as an accelerating medium. The idea behind the laser-plasma accelerator is very simple: a short intense laser (laser wakefield accelerator) or a superposition of two lasers pulses with slightly different frequencies (laser beat-wave accelerator) can be used to excite a longitudinal plasma wave that can accelerate an electron bunch to high energies. The ability of the plasma to support accelerating gradients as high as a few GeV/m has already been demonstrated experimentally [9], and even higher gradients are theoretically achievable. At the same time, a number of physical phenomena limit the energy of the accelerated electron bunch. Diffraction and depletion reduce the distance that the laser pulse can travel through the plasma. Dephasing [10] limits the distance over which the electron bunch remains at an accelerating phase of the plasma electric field and the number of beats in a beat-wave excitation of the channel.

Several schemes have been proposed to overcome diffraction. Relativistic guiding [11]–[14] relies on the energy dependence of the plasma frequency $\omega_{pe}/\gamma$ where $\omega_{pe} = 4\pi n_0 e^2/m$, with $n_0$ the unperturbed plasma density, $-e$ the electron charge, and $m$ the electron mass. The relativistic factor $\gamma = \sqrt{1+\beta^2/c^2}$, with $c$ the speed of light. The electron momentum $|p|$ will be largest where the laser pulse is most intense, and therefore the plasma frequency will be lower there, and the pulse will generate a nonlinear index of refraction which is larger at the center of the pulse than at the pulse edges. Analysis has shown [12] that, in steady state, relativistic guiding can focus a long laser pulse with frequency $\omega$ whenever the total power is greater than $P_c = 16.2(\omega/\omega_{pe})^2$ GW.

For the short pulses (of order a plasma wavelength) envisioned in many wakefield accelerators, however, relativistic guiding is substantially reduced [10]. This is due to the tendency of the ponderomotive force from the front of the pulse to push plasma electrons forward and generate a density increase which balances the relativistic mass increase. The plasma frequency then has no transverse variation and cannot optimally guide the laser pulse. Relativistically guided long laser pulses suffer from Raman forward and side-scatter instabilities [15]–[19]. This instability leads to the breakup of the pulse into small pulselets of order plasma wavelength. The utility of using these pulselets themselves for acceleration is at present unclear [18], [19].

An alternative scheme which has been investigated theoretically [1], [2], [20] and experimentally [21], [22] envisions guiding the laser pulse with a plasma density channel. The channel should have a higher density on the outside than on the inside, giving it an index of refraction which decreases from the channel axis. A fixed plasma channel is analogous to an optical fiber, and its guiding properties can be similar analyzed [21], [22]. The plasma channel can be used to guide short pulses for particle acceleration and has been theoretically studied for parabolic density variation [20], [23], [24] and for hollow channels [1], [2].

Analytic progress was obtained in some studies [20], [23], [24] by restricting the investigation to the case that the plasma is almost homogeneous over distances where the ponderomotive pressure of the laser is substantial (roughly equal to the laser spot size). In this approximation the plasma inhomogeneity can be neglected when calculating the density perturbation of the plasma induced by the laser (i.e., the wake). The plasma inhomogeneity was included when deriving the unperturbed wave equation for the laser. The analytically tractable form of the unperturbed laser eigenmodes in the parabolic channel was the primary motivation for assuming this particular transverse density profile in [23]. The detailed knowledge of the transverse eigenmodes facilitated the derivation of a coupled-mode differential equation in [23] which

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described the spatio-temporal evolution of an arbitrary transverse distortion of the laser pulse in the limit of $P/P_c \ll 1$.

The goal of this paper is to develop a general formalism for analyzing wakefield generation in the plasma without a priori assumptions on the ratio of the plasma and laser transverse gradients. The motivation (as well as the starting point for most of the calculations presented here) can be found in a previous study of the hollow channel wakefield accelerator (HCLWA) [1, 2]. In this scheme an evacuated channel in the plasma serves as an optical fiber which guides the laser pulse over many Rayleigh lengths. At the same time, the ponderomotive force of the laser excites wakefields at the surface of the channel, which extend to the center where they can be used for particle acceleration. As was first pointed out in [1] and [2], the accelerating mode of the HCLWA is transversely uniform. This uniformity is very attractive for particle acceleration since transverse inhomogeneities of the accelerating gradient introduce unwanted energy spreads and impose stringent limitations on the transverse emittance. Hence, the hollow plasma channel decouples the transverse profile of the laser pulse (which is in general nonuniform and cannot be easily manipulated) and the transverse profile of the accelerating mode. We note that the transverse profile of the accelerating wake is exactly uniform only when the wake has a phase velocity equal to the speed of light. Since the phase velocity of the wake matches the group velocity of the laser pulse, the wake has a transverse variation of order $\omega_0^2/\omega_0^2$.

The hollow channel can guide the laser pulse over many Rayleigh lengths. This paper presents detailed analytic and numerical studies of the electric fields excited by a nonevolving laser pulse. Investigation of the long range stability of the laser propagation, following the formalism of [23], is a topic for future study [25].

The previous analysis and numerical study of the HCLWA [1, 2] was mostly restricted to the case where the interface between the evacuated channel and the uniform plasma is infinitely sharp, although a general equation for the excitation of the wake in the inhomogeneous plasma in slab geometry was derived in [1, Appendix]. In this paper we explore the implications of this equation to the more realistic case of a finite thickness interface (which we refer to as the finite thickness channel wall), studied for the first time in [26]. We find that the accelerating mode of the channel, ponderomotively excited by the laser pulse, resonates with the plasma electrons inside the channel wall. Because of this local resonance, the accelerating eigenmode becomes singular at the location of the resonance. The time-averaged dissipated power is calculated, and the phenomenological decay rate of the wake is obtained. To clarify the mechanism for the wake dissipation, we analyze the temporal behavior of the electric fields inside the channel wall and find that (within the constraints of the linear theory) the transverse electric field at the resonant location grows secularly with time even after the laser pulse has gone by. This growth subsequently leads to wavebreaking and the production of energetic electrons which tap the energy of the wake. We introduce an effective quality factor $Q$ of the plasma channel and find limits on the number of the electron bunches that can be injected into the wake created by a single laser pulse. For instance, we find that for a typical case when the thickness of the channel wall is the same as its half-width, the accelerating wake decays after one oscillation.

As was mentioned in [1] and [2], the idea of the hollow channel accelerator bears some resemblance to the plasma fiber accelerator proposed by Tajima. An important difference between the two approaches is that, in the case of the plasma fiber accelerator, the plasma is overdense while in the case of the HCLWA the plasma is underdense. One of the serious shortcomings of the plasma fiber accelerator was the resonant absorption of the laser in the plasma. In the case of the HCLWA, nowhere in the plasma does the frequency of the propagating laser match the local plasma frequency. However, a more subtle effect takes place: the excited wake has a frequency lower than the plasma frequency of the surrounding plasma. This leads to the resonant absorption of the wake. While this absorption of the wake is less deleterious than the resonant absorption of the laser (since most of the dissipation occurs behind the pulse), it imposes significant constraints on accelerator design.

This paper is organized as follows. The notation and review of the basic equations governing the excitation of a plasma wake in an inhomogeneous plasma are introduced in Section II. We also review key results, previously obtained in [1] and [2], for wake excitation in a hollow channel with the step-function dependence of plasma density on the transverse coordinate. This provides the theoretical background for our study of wake excitation in the channel with finite thickness walls. Section III analyzes the eigenmodes of a hollow channel with walls of thickness $d = b\delta$, where $b$ is the half-width of the hollow channel and $\delta \ll 1$. For the case of a linear density variation in the walls, we calculate to lowest order in $\delta$ the eigenfrequency of the accelerating channel mode. We find that the eigenmode of a hollow channel with finite thickness walls is singular at the resonant location where the local plasma frequency matches the eigenfrequency of the channel. One of the consequences of the singular character of the mode is that it dissipates over time. The rate of the mode dissipation is calculated to lowest order in $\delta$ in Section IV. In reality, the singular mode would take an infinite period of time to develop and would be arrested by various nonlinear phenomena such as wavebreaking. This motivates the calculation of the temporal evolution of the fields inside the channel wall, in the framework of the linear theory, which is carried out in Section V. Much of the theoretical analysis in Sections III–V is discussed in greater detail in [26]. Section VI contains a discussion of the nonlinear evolution of the wake and the results of particle-in-cell (PIC) simulations. Our results are summarized and directions for future research are outlined in Section VII.

II. Basic Formalism and the HCLWA

A number of simplifying assumptions make the problem of wakefield generation in inhomogeneous plasma amenable to analysis. We assume that the laser pulse is much shorter than the response time of the ions, thus reducing the problem to the interaction of the intense laser with the plasma electrons in
the neutralizing background of the immobile ions. We further assume that the plasma is homogeneous in the \( z \)-direction— the direction of the laser propagation—and restrict ourselves to slab geometry, so that the plasma inhomogeneity is only a function of \( x \).

The unperturbed local plasma frequency is defined as

\[
\omega_{p0}^2(x) = \frac{4\pi e^2 n_0(x)}{m}
\]  

(1)

where \( n_0(x) \), the unperturbed plasma density, varies only in the \( x \)-direction. The plasma is taken to be underdense, so that \( \omega_{p0}^2 \ll \omega_0^2 \), where \( \omega_0 \) is the laser frequency. This allows us to separate time scales into the fast scale of order a laser period \( 1/\omega_0 \) and slow scale of order a plasma period \( 1/\omega_{p0} \). Plasma electrons move in the combined (fast scale) electric and magnetic fields of the laser and the (slow scale) fields induced by the laser in the plasma. Since the plasma frequency is assumed to be much smaller than that of the laser, we can time-average over the laser period so that electrons are only driven by the ponderomotive force of the laser.

In this paper we use a weakly relativistic approximation \[15, 16\] \( a = |eA|/mc^2 < 1 \), where \( A \) is the vector potential of the laser field. Consistent with this approximation, the calculation of the density perturbation and relativistic index of refraction will be performed to second order in the laser field, neglecting the terms proportional to \( a^3 \) and higher. For very large \( a \), the ponderomotive force will significantly distort the channel walls (blowout regime \[27, 28\]), thus precluding an analysis based on a well-defined channel wall.

The laser field is described by a vector potential \( \vec{A} \), with \( a \)

\[
\vec{A} = \frac{mc^2}{2e} \vec{a}(x, t) e^{i(k_0 z - \omega_0 t)} + c.c.
\]  

(2)

where

\[
\vec{a} \cdot \vec{c}_a = 0.
\]  

(3)

The spot size of the laser is assumed to be much larger than the wavelength \( \lambda_0 \), validating the assumption of transverse polarization in (3). It is then easy to show that the maximum particle displacement in the transverse plane is much smaller than the spot size of the laser.

Separating electron velocity into fast and slow parts, using the conservation of transverse canonical momentum, and averaging over the fast time-scale yields an equation of motion for the slow-scale plasma velocity \( \vec{v} \)

\[
\frac{\partial \vec{v}}{\partial t} = \frac{e}{m} (\vec{E} + \nabla f)
\]  

(4)

where \( f \) is the ponderomotive potential given by

\[
f = \frac{mc^2}{4e^2} |\vec{a}|^2
\]  

(5)

and \( \vec{E} \) is the electric field induced in the plasma. In deriving (4) we neglected nonlinear terms of order \( a^4 \) and higher.

As is known from earlier work \[29, 30\] on the interaction of obliquely incident \( p \) polarized lasers with inhomogeneous plasmas, calculations for stratified inhomogeneous plasmas are more easily performed in terms of the magnetic field \( \vec{B} \). Fourier transforming Faraday’s law in time and introducing a new quantity

\[
\vec{P} = \frac{i e}{\omega} \vec{B}
\]

results in

\[
\nabla \times \frac{4\pi i \vec{\sigma}}{\omega} \nabla \times \vec{P} = \nabla \times \vec{E}.
\]  

(6)

Quantities with tildes denote Laplace transform in time. Combining (6) with (4) and recalling that the current in the electron plasma is given by \( \vec{f} = e n_0 \vec{v} \) yields

\[
\nabla \times \frac{4\pi i \vec{\sigma}}{\omega} \nabla \times \vec{P} = \nabla \times \vec{E}.
\]  

(7)

Later in this section we show that the vector \( \vec{P} \) goes to zero and that the wake becomes electrostatic in a homogeneous plasma. From Maxwell’s equations

\[
\nabla \times \nabla \times \vec{E} + \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial^2 \vec{J}}{\partial t^2}.
\]  

(8)

Combining (8) with (7) we obtain

\[
\nabla \times \nabla \times \left( \frac{\nabla \times \vec{P}}{1 - \omega_{p0}^2(x)/\omega^2} + \nabla \vec{f} \frac{\omega_{p0}^2(x)}{\omega^2} \right) = \frac{\omega^2}{c^2} \nabla \times \vec{P}.
\]  

(9)

Equation (9) can be simplified to a scalar differential equation whenever the plasma has planar or azimuthal symmetry. In these instances, the outer curl in (9) can be removed and (9) is substantially simplified as follows.

*Case 1*: Slab geometry \( n_0 = n_0(x) \), \( \vec{f} = \vec{f}(x, z) \) and the TM mode is excited and \( \vec{P} = \vec{\varepsilon}_g \vec{P}(x, z) \). The TM wave is generated and \( \vec{P} = \vec{\varepsilon}_g \vec{P}(x, z) \).

The physical reason for this simplification is that from the symmetry of the cylindrical and slab geometries, the magnetic field is unidirectional. In this paper we concentrate on Case 1. The extension to azimuthal symmetry is straightforward.

By introducing \( \vec{\phi} = \vec{\varepsilon}_g \vec{P}(x, z) \), (9) is reduced to

\[
\nabla^2 \vec{\phi} + \frac{\partial \vec{\phi}}{\partial x} \frac{\partial }{\partial x} \ln \left( \frac{1 - \omega_{p0}^2(x)}{\omega^2} \right) \omega_{p0}^2(x) \vec{P} = -\frac{\omega}{c^2} \frac{\partial \vec{J}}{\partial x} \ln \left( \frac{1 - \omega_{p0}^2(x)}{\omega^2} \right).
\]  

(10)

Equation (10), in a slightly modified form, was used \[29-31\] for analyzing the resonant absorption of obliquely incident \( p \) polarized radiation by inhomogeneous plasma. With the right-hand side (RHS) vanishing, (10) was used to study electromagnetic surface eigenmodes in \[32\]. It is used for the first time, to our knowledge, for studying ponderomotively induced plasma wakes for particle acceleration.

We will concentrate on the wakes left by nonevolving laser pulses moving with speeds close to the speed of light. By
assuming all the quantities of interest to be functions of $x$ and a single longitudinal variable $\zeta = ct - z$, an arbitrary tilted quantity $\tau$ can be expressed as a Laplace transform in $\zeta$

$$\hat{\tau}(\omega, x) = \frac{d^2}{c^2} e^{i \omega \zeta/c} \tau(\zeta, x)$$

(11) thus simplifying (10) to

$$\frac{\partial^2 \hat{P}}{\partial x^2} + \frac{\partial \hat{P}}{\partial x} \ln \left( 1 - \frac{\omega^2}{\omega_{p0}^2(x)} \right) = -i \frac{\omega}{c} \hat{f} \frac{\partial}{\partial x} \ln \left( 1 - \frac{\omega^2}{\omega^2(\zeta)} \right).$$

(12)

Henceforce we will drop tildes on the variable $P$.

Equation (12) holds for any transverse density profile $\omega_{p0}^2(x)$, including those with density discontinuities. Equation (7) and a knowledge of $P(x, \zeta)$ is sufficient for computing the electric field.

Equation (12) can be solved for a continuous density profile using the boundary condition

$$\lim_{x \to \pm \infty} P(x) = 0.$$  

(13)

It is evident that $P = 0$ is the only solution satisfying (12) and (13) in a homogeneous plasma $0 = (\partial/\partial x) \omega^2_{p0}(x)$. Thus, to second order in dimensionless laser amplitude $a$, the plasma wake is electrostatic in a homogeneous plasma.

For discontinuous density profiles (e.g., a hollow channel or one with density steps), the solutions from adjacent regions of continuous density should be matched at the discontinuity. The usual technique of matching the solutions is to find invariants that remain continuous across the density jump. Close inspection of (12) gives the following continuity conditions:

$$P(x) \to \text{continuous}$$

(14)

$$\frac{\partial P}{\partial x} + \frac{i \omega}{c} \hat{f} \ln \left( 1 - \frac{\omega^2}{\omega_{p0}^2(x)} \right) \to \text{continuous}. \quad (15)$$

Equation (15) is equivalent to demanding the continuity of $E_x$, and (14) is equivalent to balancing the jump in $E_x$ with the surface charge (i.e., $\Delta E_x = 4 \pi \sigma$) with $E_x$ from (7) and $\sigma = n_0 e \times \text{surface displacement}$.

In deriving the continuity conditions (14) and (15), we have assumed that ponderomotive potential $f$ is continuous across the boundary. Since $f$ is proportional to the amplitude of laser radiation (which is guided by the channel itself), it is clear that some discontinuity of $f$ at the boundary is unavoidable if, for example, the laser is polarized in the $x$-direction. Yet, since $\omega_p^2 \ll \omega_0^2$, this discontinuity can be neglected compared to the discontinuity of $\partial P/\partial x$. Next we apply the general formalism developed here to a hollow channel.

The “eigenmodes” of an arbitrary channel of transversely inhomogeneous plasma can be found by solving (12) with $f = 0$, which then becomes an eigenvalue equation for wakes, moving with the speed of light, varying as $e^{-i(\omega/c)(ct-z)}$, where $\omega$ is a (complex) eigenfrequency which makes the left-hand side (LHS) of (12) vanish. These modes are not eigenmodes in a conventional sense because for an arbitrary density profile $n_0(x)$ (12) does not have any solutions which vary exponentially with time (very similar to Landau damped plasma waves in warm plasmas).

Analytical progress can be made for an ideal hollow channel geometry where

$$\omega_{p0}^2(x) = \begin{cases} 0 & \text{for } |x| < b \\ \omega_p^2 & \text{for } |x| > b. \end{cases}$$

(16)

We now look for the eigenfunctions $P(x)$, satisfying (12), the continuity conditions (14) and (15), at $x = \pm b$, with $f = 0$. The symmetry of the problem with respect to an $x \to -x$ transformation assures that the eigensolutions will be either even or odd in $x$. In fact, in the rest of the paper we will be only considering the odd modes since they correspond to even accelerating gradients and are excited by laser pulses which are symmetric in $x$. The even modes that are excited by an initially offset laser pulse or one that has degraded through instabilities (such as examined in [23]) can be analyzed by techniques similar to those given here.

The only eigenmode of the hollow channel is given by

$$P = \begin{cases} C \frac{x}{b} & \text{for } |x| < b \\ C e^{-b_p(x-b)} & \text{for } x > b \\ -C e^{b_p(x+b)} & \text{for } x < -b \end{cases}$$

(17)

where $k_p = \omega_p/c$ and $C$ is an arbitrary constant. The eigenfrequency of this mode, found by applying the continuity condition (15), is given by

$$\omega_{eh} = \frac{\omega_p}{\sqrt{1 + k_p^2}}.$$  

(18)

Using (7) we find the accelerating gradient inside the channel to be transversely uniform

$$E_x = \frac{C}{b}.$$  

(19)

This attractive property of the hollow channel laser wakefield accelerator was first realized in [1] and [2]. Transverse inhomogeneities in the accelerating field introduce unwanted energy spread in the bunch and impose stringent limitations on the transverse emittance. An analysis that did not approximate $v_g = c$ would find gradients of order $\omega_p^2/\omega_0^2$. Indeed, Fig. 4(a) shows a transverse profile of the accelerating obtained through PIC simulation with $\omega_0/\omega_p = 5$. A small inhomogeneity of the accelerating wake is clearly visible.

It is easy to show that the density perturbation associated with the eigenmode (17) is equal to zero inside the plasma and creates a surface charge layer at the edge of the plasma. The fields inside the channel are the fringing fields from this surface charge. By comparing the transverse and longitudinal components of the electric field in the plasma one easily finds that the trajectories of the plasma electrons are ellipses in the $x - z$ plane, with an aspect ratio equal to $\sqrt{1 + k_p^2}$.

To relate the amplitude of the wake $C$ to the strength of the laser driver $f$, the continuity condition (15) is applied to the
eigensolution (17), resulting in
\[
\frac{C}{b} = \omega_p^2 c^2 \frac{\sin \phi}{(1 + k_p b)^2 (\omega_p^2 - \omega_{ch}^2)}
\]
where
\[
\phi \approx \phi(x = b).
\]
Assuming that \(f = f(\xi, x)\), the accelerating field inside the channel can be expressed as
\[
E_x(\xi) = \frac{k_p^2}{(1 + k_p b)} \int_{-\infty}^{\xi} d\xi' \frac{k_{ch}(\xi - \xi')}{k_{ch}} \cdot \frac{df(\xi', x = b)}{d\xi'}.
\]
(22)

Note that this relation between the ponderomotive driver and the wake amplitude is identical to that in a homogeneous plasma with the small difference that \(k_{ch}\) is replaced by \(k_p\) in the sine and the coupling constant factor \((1 + k_p b)^2\) is replaced by unity. Of course, for the homogeneous plasma the ponderomotive force is evaluated on axis (i.e., at its peak), while for the channel it is evaluated at the channel wall (corresponding to a value less than the peak). Another, somewhat unexpected, consequence of (22) is that the amplitude of the wake is driven by the \textit{longitudinal} rather than transverse shape of the ponderomotive potential \(f\). Hence, despite the significant ponderomotive pressure a narrow laser pulse exerts on the plasma \textit{transversely}, it still has to be short to create substantial wakes inside the channel. As (22) indicates, the accelerating gradient inside the hollow channel is transversely uniform and determined by the amplitude of the laser pulse at the edge of the channel. Thus, the plasma channel effectively decouples the transverse profile of the laser from the transverse profile of the wake.

III. THIN-WALL HOLLOW CHANNEL

In Section II we calculated exactly the plasma wake driven by the ponderomotive force in a hollow channel. This hollow channel with a step-function density profile is clearly an idealization. In a realistic hollow channel the plasma density will rise continuously from zero (inside the channel) to a constant value far away from the channel. To quantitatively anticipate the new physics exhibited by a smooth channel, imagine that the thickness of the “channel wall” (the distance over which the density changes from zero to \(n_0\)) is much smaller than the size of the channel. We expect (and will show later in this paper) that the eigenfrequency of the channel mode, \(\omega_{ch}\), does not change significantly from that given in (18).

Since
\[
0 < \omega_{ch} < \omega_p
\]
there exists a location \(x_r\) inside the channel wall where the channel frequency matches the local plasma frequency
\[
\omega_p(x_r) = \omega_{ch}.
\]
(24)

A resonant enhancement of the electric field is expected at this location. The coefficient multiplying the first derivative of \(P\) in the eigenmode equation (12) becomes singular.

Expanding the coefficients of (12) in the vicinity of \(x = x_r\), to the lowest order in \((x - x_r)\) and defining \(L\) by
\[
\frac{d}{dx} \left( \frac{\omega_p^2}{\omega_{ch}} \right) (x = x_r) = \frac{1}{L}
\]
gives (with \(f = 0\)
\[
-P'' + \frac{P'}{x - x_r} + \frac{\omega_{ch}^2}{c^2} P \left( 1 + \frac{x - x_r}{L} \right) = 0
\]
(26)
where a prime denotes \(d/dx\).

A general solution of (26) in the vicinity of \(x = x_r\) can be expressed as a series in \(\xi = (x - x_r)\)
\[
P = D \left( \frac{k_{ch}^2 \xi^2}{2} \ln(k_{ch} \xi) + 1 + \cdots \right) + B(k_{ch}^2 \xi^2 + \cdots)
\]
(27)
where only the first terms in the power series are given, and \(D\) and \(B\) are numerical constants. Solution (27) inside the ramp region should match at the ramp boundaries with the corresponding solutions inside the channel and inside the homogeneous plasma [given by (17)].

A simple illustration of how the matching occurs is helpful: we assume, for simplicity, that \(k_p b = 1\) and \(L/b \ll 1\). Then, using (18), \(k_{ch} = k_p / \sqrt{2}\), and the boundaries of the ramp are located at \(x_r = \pm L\). To guarantee amplitude matching with solutions of (17), the constant \(D\) in (27) should be equal to the \(C\) in (17). On the other hand, the constant \(B\) has to be chosen so as to match the \textit{derivatives} of the solutions of (27) with those of (17). It is a matter of simple algebra to see that \(B = C k / L\). Notice that \(B \to \infty\) as the wall thickness \(L\) approaches zero, yet \(B\) does not contribute to the amplitude of \(P\) in this limit! The increase of \(B\) in the limit of an infinitely thin wall is related to the fact that there is a jump in transverse field \(E_x\) due to the surface charge at the edge of the plasma. As a result, for a step-function density profile, \(P\) has a discontinuous derivative at the channel edge [as seen from (17)], while for a channel with a finite-thickness wall \(P\) has a smooth maximum at \(x = x_r\).

It is evident from (7) that the axial and transverse electric fields diverge as \(x \to x_r\)
\[
E_x \propto k_{ch} C \ln(k_{ch} \xi) (k_{ch} L)
\]
\[
E_x \propto k_{ch} C (k_{ch} \xi^{-1} (k_{ch} L))
\]
(28)
(29)

The merit of using \(P\) (or \(B_p\)) to characterize the eigenmodes of an inhomogeneous plasma, instead of the electric field, is now very clear. Both components of \(E\) are divergent at the resonant location, which would make an equation for \(E\) equivalent to (12), hard to analyze even numerically. On the other hand, \(P\) stays finite and differentiable at \(x = x_r\), as seen from the asymptotic expansion (27). The robustness of \(P\) to small changes in the wall thickness suggests that the eigenfrequency of the channel \(\omega_{ch}\) does not change much from (18) if the wall thickness is small compared to the channel width.

We see that, as the channel wall becomes thinner \((L \to 0)\), for a fixed \(C\) [which corresponds to the accelerating gradient inside the channel, as seen in (19)], the amplitude of the
singularity decreases, vanishing for a step-function density profile. The electric field at \( x = x_r \) neither builds up to high values instantaneously, nor because of nonlinear effects will it ever become infinite. The dynamics of the buildup of the electric field are examined in Section V.

Before developing this time-dependent theory, we perturbatively examine the eigenmodes of a hollow channel with thin walls. In this analysis we develop an operator approach to wake excitation in an inhomogeneous plasma and perturbatively calculate the structure of the eigenmode inside the channel wall. This will show that neither the eigenfrequency, nor the accelerating mode itself, are greatly affected by the finite thickness of the channel wall. Numerical and analytical solutions to the eigenvalue equation given by the LHS of (12) are obtained for a plasma with a linearly tapered density (from zero to \( n_0 \)) over a distance short compared with the size of the channel \( b \).

By defining a linear operator

\[
\mathcal{L} = -\frac{\epsilon(x, \omega)}{\epsilon(x, \omega)} \frac{\partial}{\partial x} \left( \frac{1}{\epsilon(x, \omega)} \frac{\partial}{\partial x} \right) + \frac{\omega_p^2(x)}{c^2} \tag{30}
\]

where \( \omega \) is the frequency of the wake (not the laser field) and

\[
\epsilon(x, \omega) = 1 - \frac{\omega_p^2(x)}{\omega^2} \tag{31}
\]

(12) can be symbolically rewritten, for \( f = 0 \), as

\[
\mathcal{L} P(x) = 0. \tag{32}
\]

The frequency dependence of \( \mathcal{L} \) is suppressed for notational convenience.

It is straightforward to prove a useful identity which holds for arbitrary odd functions \( \psi_1(x) \) and \( \psi_2(x) \)

\[
\int_0^{+\infty} dx \frac{\psi_1(x)}{\epsilon(x, \omega)} \mathcal{L} \psi_2(x) = \int_0^{+\infty} dx \frac{\psi_2(x)}{\epsilon(x, \omega)} \mathcal{L} \psi_1(x). \tag{33}
\]

Thus, with the weighting factor \( 1/\epsilon(x, \omega) \), \( \mathcal{L} \) is a Hermitian operator (assuming that \( \omega \) is real). Therefore, it has a complete set of real eigenvalues \( \lambda_n \) and corresponding eigenfunctions \( \psi_n \) (that can be also chosen to be real)

\[
\mathcal{L} \psi_n(x) = \lambda_n \psi_n(x) \tag{34}
\]

where \( \lambda_n = \lambda_n(\omega) \) (since the operator \( \mathcal{L} \) is frequency dependent). Note that the lower limit of integration is chosen as zero (instead of \(-\infty\)) since, as was mentioned earlier, only odd modes (corresponding to even accelerating fields) are considered in this paper.

The orthogonality condition for the eigenfunctions is derived from (33) and (34)

\[
\int_0^{+\infty} \frac{dx}{\epsilon(x, \omega)} \psi_{n_1} \psi_{n_2}(x) = \begin{cases} 0 & \text{for } n_1 \neq n_2 \\ U_{n_1} & \text{for } n_1 = n_2 \end{cases} \tag{35}
\]

where \( U_{n_1} \) is the weighting coefficient. Index \( n \) does not imply a purely discrete spectrum; in fact, the spectrum contains both discrete (\( \lambda < k_p^2 \)) and continuous (\( \lambda > k_p^2 \)) eigenvalues. For continuous part of the spectrum, the eigenfunctions obviously are not normalizable.

The accelerating quasi-mode we are concerned with is the self-supported wake, corresponding to \( \lambda = 0 \), as explained prior to (16). The unperturbed solution for a step-function profile \( \psi_0 \) is given by (17), and its eigenfrequency \( \omega_u = \omega_{\text{ch}} \). We now assume that the thin-wall dielectric function is given by

\[
\epsilon = \begin{cases} 1 & \text{for } |x| < b(1 - \delta/2) \\ 1 - \frac{\omega_p^2(x - b - b\delta/2)}{\omega^2 b\delta} & \text{for } b(1 - \delta/2) < |x| < b(1 + \delta/2) \\ 1 - \frac{\omega_p^2}{\omega^2} & \text{for } |x| > b(1 + \delta/2) \end{cases} \tag{36}
\]

where \( \delta \ll 1 \) is the ratio of a channel thickness to the half-width of the channel. Note that the average position of the channel wall remains fixed, at \( x = b \), to exclude the changes of the eigenfrequency from variations of the channel width.

With these definitions it is straightforward to estimate, to linear order in \( \delta \), the influence of density ramp on the wake eigenmode. Equation (32) for the wake eigenfunction \( \psi \) can be rewritten as

\[
\frac{\partial}{\partial x} \left( \frac{1}{\epsilon(x, \omega)} \frac{\partial \psi}{\partial x} \right) = \frac{\omega_p^2(x)}{c^2 \epsilon(x, \omega)} \psi. \tag{37}
\]

The solutions \( \psi \) in the regions \( 0 < x < b(1 - \delta/2) \) and \( x > b(1 + \delta/2) \) are known and given by (17). Therefore, (37) can be integrated across the density ramp to yield the change in \( \psi' / \epsilon \propto E_z \). A step-function density profile has \( \delta = 0 \) and (15) is satisfied.

The function \( \psi \) (chosen to be equal to unity at \( x = x_r \)) is

\[
\psi = \begin{cases} 
\frac{A_1 x}{b(1 - \delta/2)} & \text{for } x < b(1 - \delta/2) \\
1 - \frac{B(x - x_r)^2}{A_2 e^{-k_p(x - b - b\delta/2)}} & \text{for } b(1 - \delta/2) < x < b(1 + \delta/2) \\
\frac{A_1}{b(1 + \delta/2)} & \text{for } x > b(1 + \delta/2)
\end{cases} \tag{38}
\]

where \( A_1, A_2 \), and \( B \) are constants to be determined by matching \( \psi \) and its derivatives at the boundaries. The functional dependence of \( \psi \) inside the ramp is chosen in accordance with (27); one can easily show that other terms in the expansion (27) can be neglected to linear order in \( \delta \). The plasma frequency at the resonant location

\[
x_r = b \left( 1 - \frac{\delta(k_p b - 1)}{2(k_p b + 1)} \right) \tag{39}
\]

is equal to the unperturbed channel frequency, given by (18).

Using \( \xi = x - x_r \) instead of \( x \), we find the coordinates of the inner and outer edges of the ramp

\[
\xi_{\text{in}} = -\frac{b\delta}{1 + k_p b} \tag{40}
\]

\[
\xi_{\text{out}} = \frac{k_p^2 b^2 \delta}{1 + k_p b} \tag{40}
\]

By matching the derivatives at \( \xi_{\text{in}} \) and \( \xi_{\text{out}} \) we find, to lowest order in \( \delta \)

\[
B = -\frac{1 + k_p b}{2b^2 \delta} \tag{41}
\]
and, using (41) and continuity of $\psi$, recover

$$A_1 = 1 - \frac{\delta}{2(1 + k_p b)}$$
$$A_2 = 1 - \frac{\delta k_p^2 b^2}{2(1 + k_p b)}.$$  \hspace{1cm} (42)

Expressing the plasma quantities inside the density ramp (using $\omega = \omega_{ce}$) in the new coordinates yields

$$e(\xi) = \frac{(1 + k_p b)\xi}{b^2}$$
$$k_p^2(\xi) = k_p^2 \frac{\xi - \xi_{in}}{b^2}.$$  \hspace{1cm} (43)

Substituting (43) into (37) and integrating it across the ramp gives

$$\frac{-k_p A_2}{1 - \omega_p^2/\omega^2} = \frac{A_1}{b(1 - \delta/2)}$$
$$= -\frac{k_p^2}{1 + k_p b} \int_{\xi_{in}}^{\xi_{out}} \frac{d\xi}{\xi} \frac{(1 - B\xi^2)(\xi - \xi_{in})}{(1 - B\xi^2)(\xi - \xi_{in})}. \hspace{1cm} (44)$$

The integral on the RHS of (44) has a singularity at $\xi = 0$ and must be taken in the principal value sense.

After some algebra, assuming $\omega = \omega_{ce} + \Delta \omega$, we find, to linear order in $\delta$

$$\frac{\Delta \omega}{\omega_p} = \frac{k_p^2}{b(1 - \delta/2)} \left[ (1 - k_p^2 b^2/2 - k_p b \ln k_p b) \right]/(1 + k_p b)^{3/2}.$$  \hspace{1cm} (45)

The perturbed channel frequency is thus a function of both the width of the channel wall and the average width of the channel.

The dependence of the frequency shift on the channel width $b$ and channel wall thickness $\delta$ was also found by solving numerically the eigenvalue equation (37) and is shown in Fig. 1, where $\Delta \omega/\omega_p$ is plotted as function of $\delta$ for wall boundaries given by: i) $x_{in} = b(1 - \delta), x_{out} = b$, ii) $x_{in} = b, x_{out} = b(1 + \delta)$ and iii) $x_{in} = b(1 - 0.5\delta), x_{out} = b(1 + 0.5\delta)$. In all three cases $k_p b = 1$. Note that the frequency shift is, as expected, almost equal to zero in case iii), since $\Delta \omega$ given by (45) vanishes for $k_p b = 1$. We believe that the frequency shift in case iii) did not vanish identically due to a loss of numerical accuracy in the vicinity of the resonance point $x_r$. Case i) gives a frequency shift almost identical to the shift that corresponds to making a step-function channel narrower by a factor $(1 - \delta/2)$, while case ii) corresponds to making a step-function channel wider by a factor $(1 + \delta/2)$.

As (45) indicates, for typical parameters with $k_p b \approx 2$ the relative frequency shift of the wake mode due to a 10% wall thickness is about 1%. Hence, our intuition is correct and the (real) frequency of the wake excited inside a smooth wall channel is very robust to variations in thickness of the channel wall (as is the structure of the accelerating field inside the channel). Yet the gradual density ramp of the plasma introduces an imaginary part to the frequency of the channel. In the next section we calculate it perturbatively by evaluating the energy dissipated inside the density ramp per cycle of the wake oscillations.

![Fig. 1](image)

**Fig. 1.** Normalized frequency shift as a function of a fractional wall thickness, for three different positions of the channel walls. In all cases $k_p b = 1$. Case (i) $x_{in} = b(1 - \delta), x_{out} = b$. Case (ii) $x_{in} = b, x_{out} = b(1 + \delta)$. Case (iii) $x_{in} = b(1 - 0.5\delta), x_{out} = b(1 + 0.5\delta)$.

### IV. DISSIPATION OF THE WAKE

A simple observation helps to understand why the surface modes in the hollow plasma channel are dissipative. As we have noted in the previous section, there exists a location, $x_r$, inside the channel wall where the surface mode frequency matches the local plasma frequency. At this location the surface mode can mode-convert into the plasma wave. Since the plasma waves in the inhomogeneous plasma form a continuum, dissipation of the surface mode ensues. The effect of coupling to a continuous spectrum of modes was known for many years—one of the well-known examples of this phenomenon is Alfvén heating, where a compressional (fast) Alfvén wave dissipates by coupling to a continuum of shear (slow) Alfvén waves [33], [34].

To calculate the rate at which the energy is dissipated inside one of the channel walls, the time averaged product of $\vec{E} \cdot \vec{j}$ is integrated over the volume of the wall

$$Q_d = \frac{b_x b_y}{2} \frac{b_x b_y}{b_y} \Re \int_{b(x=0/2)}^{b(x=b/2)} dx \frac{\partial^2 \psi}{\partial x^2} \vec{E} \cdot \vec{j} \hspace{1cm} (46)$$

where $b_x$ and $b_y$ are the dimensions of the wall (which will cancel out when the rate of the decay is computed). Using (6) and (7) for the current and the electric field results in

$$Q_d = -\frac{b_x b_y \omega_{ce}}{8\pi} \Im \int_{b(x=0/2)}^{b(x=b/2)} dx \frac{\partial^2 \psi}{\partial x^2} \left[ \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial z} \right] - \frac{1}{1 - \nu^2(x)/\omega_{ce}^2 + i\nu^2}$$  \hspace{1cm} (47)

where $\nu$ is an infinitesimal positive quantity having the dimension of frequency, which is necessary to insure causality.

It is easy to convince oneself that the second term in the integrand of (47) dominates the first term in the limit of $\delta \ll 1$ (physically, this corresponds to the fact that the $x$-component of the electric field is much larger at the resonant location than the $z$-component; hence the dissipation due to $j_x E_z$ dominates the dissipation due to $j_z E_z$). Since the imaginary part of the integral in (47) comes from the singularity in the dielectric coefficient, the transverse dependence of $P$ can be neglected...
and taken out of the integral. Using $\xi = x - x_r$ instead of $x$ results in

$$Q_d = \frac{b_p b_y \omega_{ch}^3}{8 \pi c^2} \frac{|C|^2}{1 - \frac{1}{\tau}} \int_{x_{in}}^{x_{out}} \frac{d\xi}{i \tau - (1 + k_p b) \xi / (a b)}$$  \hspace{1cm} (48)$$

where $\tau$ is an infinitesimal positive number. Evaluating (48) yields

$$Q_d = \frac{|C|^2 \omega_{ch}^2 k_p^2 b_p b_y b_y}{8 (1 + k_p b)}.$$  \hspace{1cm} (49)$$

The source of the energy dissipated in the channel walls is the stored energy of the wake, which can be easily calculated by summing the electric, magnetic, and particle kinetic energies inside the channel and in the bulk of the plasma. To zeroth order in $\delta$ this energy is given by

$$W = \frac{|C|^2 b_x b_y}{16 \pi b} \left( 1 + \frac{2 (k_p b)^2}{3 (1 + k_p b)} + \frac{k_p b}{2 (1 + k_p b)} \right) \left( 1 + \frac{(2 + k_p b)^2}{(k_p b)^2} \right).$$  \hspace{1cm} (50)$$

Another important characteristic of the wake is the axial projection of the Pointing vector given by

$$S_x = \frac{b_y b_x}{8 \pi} \int_0^{\infty} \, dx \, B_x^* E_x.$$  \hspace{1cm} (51)$$

Evaluating (51) yields

$$S_x = \frac{|C|^2 b_x b_y}{16 \pi b (1 + k_p b)} \left( 2 k_p b^2 - 1 \right).$$  \hspace{1cm} (52)$$

Note from (52) that for $k_p b = 1$ the Pointing flux points in the negative $x$-direction. Group velocity of the wake perturbation can be evaluated as

$$\frac{v_g}{c} = \frac{S_x}{W}.$$  \hspace{1cm} (53)$$

For $k_p b = 1$ we find $v_g / c \approx -0.04$. This is in contrast with the wake excited in a cold homogeneous plasma which has a vanishing group velocity.

The time-averaged energy conservation can be now expressed as

$$\frac{\partial W}{\partial t} + \frac{\partial S_x}{\partial z} = -Q_d$$  \hspace{1cm} (54)$$

yielding the imaginary part of the surface mode

$$\omega_{im} = -\frac{dW / dt}{2W + 2S_x / c}.$$  \hspace{1cm} (55)$$

Substituting the rate of dissipation $dW / dt$ from (49) and the wake energy $W$ from (50) and evaluating (55) for $k_p b = 1$ we obtain

$$\omega_{im} = 0.25 \omega_{ch}.$$  \hspace{1cm} (56)$$

Hence, for $k_p b = 1$ and $\delta = 1.0$, one expects the amplitude of the wake to decay by about 70% after one oscillation. We note that the same result can be obtained by solving the dispersion relation for complex $\omega$ in analogy to the method in [32].

An important question to examine is where this energy goes. Since no collisions were assumed to dissipate the energy, one cannot attribute the loss of energy by the wake to bulk heating of the plasma. And, in fact, the singular fields that characterize the mode would take infinitely long to build up and would clearly violate the assumptions of our linear theory. In the next section we examine time evolution of the electric field inside the channel wall, excited by a ponderomotive force of an intense laser pulse.

V. TIME-DEPENDENT EXCITATION OF THE WAKE

After a detailed study of the properties of the channel eigenmodes in Section III, we now examine how these wakes are excited by the ponderomotive force of the laser. To do that we rewrite (12) in the operator form

$$LP = -i \frac{\omega}{c} \frac{\partial \ln \epsilon(x)}{\partial x}.$$  \hspace{1cm} (57)$$

The function $P$ can be expanded in a complete set of eigenmodes $\psi_n$ of the operator $L$ given by (34)

$$P(x, \omega) = \sum p_n(\omega) \psi_n(x, \omega).$$  \hspace{1cm} (58)$$

One of the important conclusions of Section III is that the eigenfunctions $\psi$ and the corresponding eigenvalues are only slightly perturbed by the finite thickness of the channel wall. Hence, to a high degree of accuracy, the $\psi_n$'s can be chosen to be the eigenfunctions (and $\lambda_n$ the eigenvalues) of (32) with a step-function density profile and $\omega^2 p(x)$ given by (16). Using the orthogonality conditions (35), the expansion coefficients can be expressed as

$$p_n = \frac{\omega}{\lambda_n \epsilon_n} \int_0^{\infty} \, dx' \, f(x', \omega) \psi_n(x', \omega) \frac{\partial}{\partial x'} \left( \frac{1}{\epsilon(x', \omega)} \right).$$  \hspace{1cm} (59)$$

By examining the spectrum of (34) with a step-function density profile one can demonstrate that the only eigenvalue function $\lambda_n(\omega)$ that has a pole for any complex $\omega$ is the one with the discrete spectrum eigenvalues $\lambda_0(\omega)$, which vanishes at $\omega = \pm \omega_{ch}$. Alternatively, from the results of Section II, a plasma channel with infinitely sharp walls supports only one eigenmode (these self-supported modes correspond to poles of eigenvalue functions).

From (7), (58), and (59) we can find the induced accelerating field inside the channel. Since $\lambda_0(\omega)$ has a pole at $\omega = \pm \omega_{ch}$, we need to find its behavior in the vicinity of the pole. Since $\lambda_0(\omega)$ is regular in the vicinity of the pole, we consider $\omega > \omega_{ch}$, which, as we prove later, is equivalent to $\lambda > 0$.

Solutions to (34) are then given by

$$\psi = \begin{cases} \sin \lambda^{1/2} x & \text{for } x < b \\ \sin \lambda^{1/2} b & \text{for } x > b \end{cases}$$  \hspace{1cm} (60)$$

where we have dropped the subscript on $\lambda$. Applying the continuity condition (15), with $f = 0$, yields an implicit $\lambda(\omega)$

$$\tan(\lambda^{1/2} b) = \frac{\omega^2}{\lambda^{1/2} b} - 1$$  \hspace{1cm} (61)$$

where

$$\frac{\omega^2}{\lambda^{1/2} b} = \frac{\omega^2}{k_p b (k_p^2 - \lambda)^{1/2}}.$$  \hspace{1cm} (62)$$
Linearizing (61) in the vicinity of $\omega = \pm \omega_{ch}$ and $\lambda = 0$ yields
\[
\lambda = \frac{(1 + k_p b)^2}{k_p b - 1/2 + k_p^2 b^2/3} \frac{\omega^2 - \omega_{ch}^2}{c^2}.
\]
We also compute the normalization constant
\[
U_0 = \int_{-\infty}^{\infty} dx \ \frac{\psi_0^2}{\epsilon}
\]
which, for $\omega = \pm \omega_{ch}$, is equal to
\[
U_0 = -\frac{1/2 + k_p^2 b^2/3}{k_p b}.
\]

The accelerating gradient inside the channel is computed by observing that inside the channel $\omega_n^2(x) = 0$, so that the first term of (7) can be combined with (59) to obtain
\[
E_z(x, \zeta) = \int \frac{d\omega}{2\pi} e^{-i(\omega \zeta)(\zeta - \zeta')} \sum_n \frac{\psi_n^*(x, \omega)}{U_n \lambda_n(\omega)} \frac{\psi_n(x', \omega)}{\epsilon(x', \omega)} \left( \frac{1}{\epsilon(x', \omega)} \right) \frac{\partial f(x', \zeta')}{\partial \zeta'}
\]
where the first integration is carried out along a contour in complex $\omega$ plane, above the real axis, to ensure causality. Even though this calculation appears to be much more cumbersome than the one used in Section II, it is worthwhile to prove that (65) yields the same result as (22) for a channel with infinitely thin walls (i.e., $\epsilon$ changes as a step-function). The method of modal expansion, while almost being “too powerful” for finding $E_z$ inside the channel with $\delta = 0$, is the only method for finding the temporal evolution of the laser-driven fields in the channel with finite wall thickness.

As mentioned earlier, only the lowest eigenmode of the discrete spectrum has an eigenvalue function $\lambda(\omega)$ with a pole. Thus, after closing the integration contour in the lower half of complex $\omega$ plane, summation in (65) contains only one nonvanishing term—the lowest discrete mode. Since the residues at the pole of $\lambda(\omega)$ at $\omega = \pm \omega_{ch}$ were computed earlier, we substitute (62) and (64) into (65), and, adding the two residues, recover (22). Note that the second integral in (65) reflects the previously mentioned fact that the channel mode is driven by the longitudinal profile of the laser pulse at the plasma discontinuity [see (22)], where the derivative of the index of refraction is the largest.

The second integral in (65) points to an important difference between the case of an infinitely sharp boundary and finite width wall. In the former case, the $x'$ derivative in this integral becomes a delta function, and the integral itself reduces to a number. In the latter case, this integral contains a line of poles in $\omega$ plane corresponding to zeros of the dielectric function $\epsilon$ at different transverse locations inside the plasma wall. This situation is formally similar to the one that exists in a warm plasma, where resonant denominators of the form $(\omega - k_v)$ assume the role of the dielectric constant here. The branch cut along the continuous line of poles in complex $\omega$ plane leads to an algebraic (as a function of $\zeta$) decay of the fields, similarly to other physical problems where the natural modes of oscillation have a finite support [35].

We can now establish how the fields grow in time at the resonant location $x = x_r$. Formally, one expects the field to grow indefinitely since the field of the wake eigenmode diverges near resonant location [see (27)]. An unusual physical situation occurs: as the laser pulse propagates through the plasma, it generates a wakefield, which, near the resonant point, continues to grow indefinitely (in the linear theory, of course) after the laser pulse is gone!

As (7) indicates, the total electric field inside the plasma is given by a sum of two contributions. The first term on the RHS of (7), at a given location $x$, has a simple pole in the $\omega$-plane and, thus, cannot lead to a (formally) infinite growth of the field. Physically, the first term represents an oscillation, excited by the laser pulse, at the local plasma frequency $\omega_p(x)$. Thus, we can neglect this term and compute the contribution of the second term on the RHS of (7).

Similarly to (65), a modal expansion for $E_x$ at an arbitrary location in plasma gives
\[
E_x(x, \zeta) = -\int \frac{d\omega}{2\pi} e^{-i(\omega \zeta)(\zeta - \zeta')} \sum_n \frac{\psi_n^*(x, \omega)}{U_n \lambda_n(\omega)} \frac{\psi_n(x', \omega)}{\epsilon(x', \omega)} \left( \frac{1}{\epsilon(x', \omega)} \right) \frac{\partial f(x', \zeta')}{\partial \zeta'}
\]
where $\theta = \frac{\partial f(x', \zeta')}{\partial \zeta'}$.

We now calculate the contribution of $n = 0$ mode can contribute to indefinite growth of the electric field at $x = x_r$. By retaining only the $n = 0$ mode, and setting $x = x_r$, observe that the integrand in (66) has a double pole at $\omega = \pm \omega_{ch}$. We thus obtain
\[
E_x(x_r, \zeta) = \frac{k^2 b}{1 + k^2 b} \int_{-\infty}^{\infty} d\zeta' G_{\epsilon}(\zeta' - \zeta') \frac{d^2 f(\zeta', x = b)}{d\zeta'^2}
\]
where $G_{\epsilon}(\zeta - \zeta')$ is a Green function of the transverse electric field at resonant point given by
\[
G_{\epsilon}(\zeta) = \left( \frac{\zeta}{2} \right) \cos(k_v \zeta) + \frac{\sin(k_v \zeta)}{k_v}.
\]
The wake has the temporal behavior we expected—it grows indefinitely with time.

Similarly, one can estimate that the axial field at $x_r$ will be growing logarithmically with time. If $E_{x0}$ is the transverse field at the edge of the channel, the amplitude of the field at $x = x_r$ roughly grows as
\[
E_x(x_r) = \frac{k_{ch} C}{2} E_{x0}.
\]

Thus, after the laser passes through the channel, electric field inside the ramp region continues to grow in the absence of an external driver.

The situation here is (at least formally) very similar to the excitation of a linear plasma profile by an external capacitor. An inhomogeneous plasma with a linear density ramp which is subject to a sinusoidally varying (with time)
electric field produced by a plane capacitor is analyzed in [36]–[39]. The electric field is colinear with the density gradient, and its frequency matches the local plasma frequency at the resonant location \( x = x_r \). An obliquely incident \( p \)-polarized electromagnetic wave can also play the role of a capacitor. Secular growth of the electric field at resonant point, which saturates through collisions, relativistic corrections, wavebreaking, particle ejection, etc., was predicted [36], [37]. The important difference between the hollow channel and the excitation of an inhomogeneous plasma with a capacitor is that no external capacitor is needed in our case. Instead, a hollow channel mode is excited, which serves as a capacitor to drive the internal modes of inhomogeneous plasma.

VI. NONLINEAR EVOLUTION AND NUMERICAL RESULTS

In this section we give a scenario of a nonlinear evolution of the wake which is confirmed by numerical simulations.

Initially, the fields inside the channel and in the inhomogeneous plasma region are not influenced by the dramatic growth of the field at resonant point. But later, as \( E_z \) builds up, wavebreaking [40] occurs, energetic electrons are ejected into the channel, thereby dissipating the wake. The process of wavebreaking and electron acceleration (in the direction of decreasing plasma density, that is, into the channel) has been extensively discussed in the literature [36]–[39].

The mechanism for wavebreaking in the limit of \( r_0 \ll \text{bb} \) was discussed in the literature [36], [39]. Briefly, the spatio-temporal structure of the field near resonance point is such that it has the form of a wave packet, moving with a characteristic phase velocity \( v_{ph} \approx \omega_{ch} \Delta L \), where \( \Delta L \) is the size of a resonance region. Wavebreaking occurs when electron velocity (which grows linearly with time) matches \( v_{ph} \) (which decreases linearly with time). The number of oscillations before the wave breaks is roughly given by

\[
N = \frac{\omega_{ch}}{\omega_{im}} \approx (ab/r_0)^{1/2}
\]  
(70)

(see [38]). In the case of \( r_0/(ab) \approx 0.01 \), wavebreaking was numerically observed (see below) to occur after the first 1.5 oscillations, in good agreement with (70).

It is important to emphasize that it is not the wavebreaking inside the channel wall per se which causes the decay of the accelerating wake. Wavebreaking is only a mechanism for transferring the energy from the wake to the energetic electrons, and the rate at which this transfer occurs clearly depends on the intensity of the electric field at the resonant location. The effective \( Q \) of the plasma channel is given by

\[
Q = \frac{\omega_{ch}}{\omega_{im}}
\]  
(71)

where \( \omega_{im} \) is the imaginary part of the mode given by (55) and quantified for a particular case of \( k_p b = 1 \) by (56).

To test the mode structures and the damping rate of the mode, we perform two-dimensional fully electromagnetic PIC simulations of the hollow channel laser wakefield using the code ISIS [41] modified to move with the laser beam. The simulations are in slab geometry (coordinates \( x, z, v_x, v_y, v_z \)); the parameters used in the simulations are given in the caption to Fig. 2. Fig. 2 shows the axial field component \( E_z \) of the wake in a sharp boundary channel. The slight damping of the field amplitude may be numerical, due to the finite cell size. Fig. 3 shows the \( E_z \) field along the channel axis with \( \delta = 1.0 \). The field amplitude decays from 0.012 to 0.03 after the first cycle which is in good agreement with (56), although the assumption \( \delta \ll 1 \) is not satisfied. We have performed other simulations with smaller values of \( \delta = 0.5 \) and 0.3. The numerically obtained damping rates in the simulations are somewhat larger than that predicted by (56) (although they increase monotonically with \( \delta \)). We believe that the discrepancy is due to the fact that in a channel with thick walls, the wakefield quickly becomes nonlinear at the resonant location, steepens, and eventually breaks. Therefore, one should not expect a perfect agreement of the linear model with the fully nonlinear simulations. The agreement in the case of \( \delta = 1.0 \) is more fortuitous than anticipated.

To verify the linear scenario of a temporal buildup of the transverse fields at the resonant location, we show in Fig. 4 the field structure of the wake at an early stage. The
uniform transverse profile of $E_x$ and linearity of $E_x$ inside the channel resembles the mode structure in a sharp boundary case. Nevertheless, in a sharp boundary channel, the wake modes can exist for a long time without distortion. In a finite wall channel, the resonant absorption in the wall damps the wake quickly. As is shown in Fig. 5, both $E_z$ and $E_x$ at resonant location become much larger than the corresponding fields inside the channel. By comparing Fig. 5(a) and (b), it is clear that the $E_x$ component dominates the energy absorption of the wake as was pointed out in Section IV.

Dissipation of the accelerating mode imposes serious limitations on the operation of future laser wakefield accelerators. It introduces an effective $Q$ of the plasma channel, which can be quite low (as seen in Fig. 3) and prevents accelerating multiple bunches by the wake created in a single laser shot. It appears that to increase $Q$ one has to use channels with a sharp boundary, as suggested by (56). Nonlinear dissipation of the wake in a channel with $b \delta \geq r_0$ is being presently investigated.

VII. CONCLUSION

In this paper we analyzed wake excitation in an inhomogeneous plasma. A general formalism for excitation of the accelerating mode in nonuniform plasma was developed, and a hollow channel accelerator with a step-function density profile [1], [2] was considered as an example. For the first time a more realistic case of a channel with a smooth interface between vacuum and plasma was considered. We have proven that the structure of the accelerating mode is only slightly perturbed by the finite thickness of the channel walls. The correction to the real part of the eigenfrequency of the accelerating mode due to finite thickness of the channel was obtained analytically and numerically and shown to be small. The imaginary part of the eigenfrequency was obtained by calculating the rate of energy dissipation inside the channel wall.

Ponderomotive excitation of the wake by an intense laser pulse was considered as a function of distance from the head of the laser pulse. It was shown analytically that at the location inside the channel wall, where the local plasma
frequency matches the channel eigenfrequency, the transverse electric field is driven resonantly. A Green’s function for the transverse electric field at the resonant location was derived, predicting a linear growth of the amplitude of the field with time. A somewhat exotic scenario was predicted, in which the accelerating wake is dissipated because the transverse electric field at the resonant location grows secularly (in linear theory) after the short laser pulse (which excited the wake) has already propagated downstream. Nonlinear effects, most notably wavebreaking and production of energetic electrons, which can arrest the growth of the electric field, were discussed and verified by PIC simulations. An important implication is an effective quality factor $Q$ for the channel accelerator, and a limit on the number of electron bunches that can be accelerated in the wake of a single laser pulse. An important continuation of this work will be to carry out PIC simulations for different thicknesses of the channel wall (of order of the excursion of a surface electron, or smaller) to identify the most promising regime of operating the hollow channel laser wakefield accelerator.

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