Comparison of the Laser Wakefield Accelerator and the Colliding Beam Accelerator

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Abstract. With the advent of chirped pulse amplification and related technologies, much research has been devoted to laser pulse shaping for optimal wake generation in a plasma based accelerator. Also, there has been a recent proposal for a colliding beam accelerator (CBA), which uses a detuned pump laser to enhance a standard LWFA wake. We use analytic scalings and PIC simulations to illustrate optimal wake generation in the LWFA under constraints of maximum laser energy, intensity, and bandwidth. We then compare the optimized LWFA to the CBA, finding that while the addition of a pump will increase the wake of a single pulse, the CBA is inferior to a single- or multiple-pulse LWFA of identical total laser energy.

There have been several proposed concepts for using high power lasers to drive large amplitude, high phase velocity Langmuir waves suitable for particle acceleration in plasma (see, e.g. [1], and references therein). Two among these that rely on femtosecond pulses are the laser wakefield accelerator (LWFA) [2] and the recently proposed colliding beam accelerator (CBA) [3]. As chirped pulse amplification (CPA) [4, 5] and related optical techniques [6] increase laser power and our ability to precisely time, shape, chirp, and control the spectral content of pulses, it is natural to try to determine how suitably shaped laser profiles may optimally produce wakefields. We can consider this problem in terms of maximizing wake amplitude, efficiency, or some other measure of performance under appropriate physical constraints such as laser power, energy, or bandwidth.

In what follows, we use the peak longitudinal electric field as our chief figure of merit, primarily because it is the potential for high acceleration gradients that most strongly motivates plasma-based accelerator research. Additionally, when the acceleration length is limited by the dephasing length (as is typical), maximizing the peak electric field is equivalent to maximizing the achievable particle energy gain. Using maximum wake amplitude as our criterion, we summarize LWFA optimization, and then compare the LWFA findings to the less familiar CBA under constraints of fixed laser amplitude and/or energy.

1. OPTIMIZATION OF THE LWFA

The LWFA uses a short, intense laser pulse of width \( \sigma \sim \omega_p^{-1} \) to ponderomotively excite a large amplitude longitudinal electric field with near-luminal phase velocity. Assuming a one-dimensional, tenuous plasma (\( \omega_p^2 \ll \omega_0^2 \)), the wake amplitude \( \phi \equiv e\Phi/mc^2 \) created
by a quasistatic laser with dimensionless vector potential $a \equiv eA/mc^2$ is

$$\frac{\partial^2 \phi}{\partial \zeta^2} = \frac{\omega_p^2}{2} \left[ \frac{1 + |a|^2}{(1 + \phi)^2} - 1 \right],$$

(1)

where $\zeta \equiv t - z/c$ is the co-moving coordinate. In the linear regime $|a|^2 \ll 1$, the right hand side of (1) becomes $\omega_p^2 |a|^2/2$, and many of the optimization results follow by considering $\phi$ to be a driven simple harmonic oscillator. For lasers of arbitrary amplitude, the equation for $\phi$ is an oscillator with nonlinear forcing, for which many of the general conclusions drawn from the linear case still hold.

When constrained by a fixed maximum amplitude $|a| \leq a_{\text{max}}$, the largest linear wake is produced by a periodic train of short square pulses spaced at the plasma period $\lambda_p/c$, each of duration $\lambda_p/2c$. The nonlinear case has a similar solution consisting of square pulses, with pulse duration and spacing governed by the effective nonlinear plasma period [7]. Restricting ourselves to the linear regime, we find that for a single pulse with gaussian profile $|a|^2 = a_g^2 e^{-\zeta^2/\sigma^2}$, the optimal wake has $\sigma \omega_p = 2$, for which we find

$$E_c = E_0 \sqrt{\frac{\pi}{4e}} a_g^2 \approx 0.6 E_0 a_g^2,$$

(2)

where $E_0 \equiv mc\omega_p/e$ is the plasma wavebreaking field.

For laser pulses constrained solely by their total energy, the story is somewhat different. In the linear regime, a delta function pulse $|a(\zeta)|^2 = a_g^2 \delta(\zeta)$ maximizes the wake produced for a fixed laser energy [8]. Although a delta function profile cannot be produced by any finite-bandwidth laser, it can also be shown that given a fixed total energy, plasma response is maximized for pulses that are as short as the laser bandwidth permits. The optimization results are essentially the same in the nonlinear regime, although in this case multiple narrow pulses can generate a larger amplitude wake than a single pulse of identical total energy. Finally, if one considers the more physically realistic case of either a slab-symmetric or axially-symmetric plasma channel, the linear results are essentially the same as those previously quoted: for a fixed total energy, plasma wake increases as the pulse shape approaches a delta function.

To illustrate the differences obtained when maximizing the wakefield amplitude under the constraints of either fixed laser intensity or fixed total energy, we plot in Figure 1 the single pulse theoretical predictions and simulation results from the PIC code 1d-XOOPIC [9]. For a laser of fixed amplitude, Figure 1(a) demonstrates that both theory and XOOPIC predict “resonant” pulses with $\sigma \omega_p \sim 1$ will most effectively drive plasma wakes. On the other hand, Figure 1(b) shows that a pulse of fixed total energy produces an increasing plasma response as it become more narrow and intense for all but the most narrow of pulses. In this case, the envelope approximation breaks down, but if both the envelope and the carrier frequency are included in (1), agreement between theory and the simulation can be restored. In this way, Figure 1(b) indicates that increasingly narrow pulses of identical total energy produce greater wakefields provided that $\sigma \omega_p \geq 2\lambda$. Additionally, calculations involving these short pulses must take care that their time averaged electric field integrates to zero, as explained in the Appendix. Finally, we note that simulation and theory for linear pulses (not shown) demonstrate very similar results.
2. WAKE GENERATION IN THE COLLIDING BEAM ACCELERATOR

The colliding beam accelerator (CBA) is realized using two counter-progating lasers: a short seed pulse whose duration is of order $\omega_p^{-1}$, and a long pump beam, detuned in frequency from the seed pulse by $\Delta \omega \sim \omega_p$. Assuming a tenuous plasma, so that $\omega_0$, the frequency of the seed pulse, satisfies $\omega_0/\omega_p > \Delta \omega$, the analysis of plasma response was given in Ref [3] by considering the equation of motion for a single electron. Defining the dimensionless vector potentials $a_0$ and $a_1$ for the seed and pump lasers, and switching to the co-moving coordinate $\zeta = t - z/v_g$, the equation of motion for the phase of the $j$th electron $\psi_j = 2k_0z_j - \Delta \omega_j$ is found to be

$$\frac{\partial^2 \psi_j}{\partial \zeta^2} + \omega_B^2 \sin \psi_j = -\omega_p^2 \sum_{l=1}^{\infty} n_l e^{il\psi_j} - \frac{2e\omega_0}{mc} E_{\text{fast}} + c.c.,$$

where $\omega_B^2 = 4\omega_0^2 a_0 a_1$ is the bounce frequency in the pondermotive bucket.

From (3) we can identify two distinct plasma waves generated by the lasers. The first wave arises from the sum over density harmonics $n_l$, predominantly involving photon deceleration associated with the pondermotive beating of the two lasers. Because of its slow phase velocity ($v_p/c = \Delta \omega/2\omega_0 \ll 1$) this plasma wave is unsuitable for electron acceleration (but may prove useful for accelerating ions [10]).

The second plasma wave $E_{\text{fast}}$ has a non-zero average over the electron phase; its importance was previously noted in [11] for the study of Raman backscatter. This second wave is dominated by photon exchange between the lasers which, to conserve momentum, deposits a net momentum in the plasma. Because of its slow phase velocity ($v_p/c = \Delta \omega/2\omega_0 \ll 1$) this plasma wave is unsuitable for electron acceleration (but may prove useful for accelerating ions [10]).

The electron current, in turn, generates a plasma wave via Faraday’s law $\partial_t E_{\text{fast}} = -4\pi \langle J \rangle$. Because this wave has a phase velocity equal to the seed pulse’s group velocity, it can provide useful electron acceleration.
The CBA as envisioned by [3] has two significant attractive characteristics: its potential applicability as a multi-staged plasma linac and its capability to create enhanced acceleration gradients for a given laser intensity. The former relies on manipulation of the plasma wave phase through the use of multiple pumps, each appropriately timed and detuned from the seed pulse. While this does present the interesting possibility of a staged accelerator unlimited by the dephasing length, we will not consider it further here. Instead, we focus on the second feature, and compare the plasma response of the CBA and LWFA under realistic constraints.

2.1. The Linear Regime of the CBA

If the slow wave is linear, the harmonics \( n_l \) for \( l > 1 \) in (3) can be dropped, and one can find a closed form for the accelerating wakefield. By choosing a gaussian seed pulse of width \( \sigma \omega_p = 2 \), and detuning \( \Delta \omega = \pm 1.1 \omega_p \), the peak electric field reaches a maximum value

\[
E_{fast} \approx 0.6 E_0 \frac{16a_1^2 \omega_0^3}{\omega_p^3} a_0^2. 
\]  

(4)

In this case, the addition of the pump multiplies the optimized single pulse LWFA longitudinal field (2) by an additional factor of \( 16a_1^2(\omega_0/\omega_p)^3 \), and the CBA outperforms the single pulse LWFA whenever \( 4a_1 > (\omega_p/\omega_0)^{3/2} \). For this reason, it is apparent that the CBA might effectively use low intensity lasers (\( a_0, a_1 \sim .004 - .02 \)) to achieve plasma wakefields \( \sim 1 \text{ GeV/m} \) in the linear regime. One should also note that the accelerating

![Figure 2](image-url)
wakefield is accompanied by a large amplitude, high frequency, slow plasma wave. Because this slow plasma wave averages to zero over the phase of the electrons, it should not directly affect particle acceleration.

To illustrate an enhanced wake, we have simulated the CBA with a density $n_0 = 10^{18} \text{cm}^{-3}$ and $a_0 = a_1 = 0.02$ using 1d-XOOPIC. We plot the longitudinal field in Figure 2. The seed pulse propagates to the left, colliding with the pump at $k_p z \approx 30$, so that for $8 < k_p z < 25$, one can see both the accelerating wake (wavelength $2\pi c/\omega_p$) and the superimposed slow wake (wavelength $\pi c/\omega_0$). Figure 2 indicates an enhanced accelerating wake $E_{\text{fast}} \approx 1\text{GeV/m}$, which is about 43 times greater than that for the single pulse LWFA. In this way, an LWFA experiment would require a laser with intensity $4.5 \times 10^{16} \text{W/cm}^2$ in place of the CBA simulated $10^{15} \text{W/cm}^2$ laser to achieve the same plasma response. This enhancement is, however, considerably less than the factor of $\approx 235$ increase predicted by (4). Presently, it is unclear if a more complete analysis of (3) will give greater agreement; based on simulation results over a wide range of system parameters, we will consider the enhancement formula (4) as an upper bound for the linear CBA wake.

2.2. Particle Trapping Regime for the CBA

As one increases the laser driver intensity, larger wakes are possible, but the linear analysis of Section 2.1 no longer holds. For sufficiently intense lasers, one has $\omega_B^2 = 4\omega_0^2 a_0 a_1 \gg \omega_p^2$, so that the right hand side of (3) becomes relatively unimportant. In this case, electrons can become trapped in the ponderomotive bucket, where they can acquire...
an average momentum if the seed pulse duration and detuning is appropriately chosen. In the case where \( \omega_p^2 \gg \omega_o^2 \), one can estimate the average imparted momentum, which produces an accelerating wake amplitude

\[
E_{fas} \approx 2E_0(a_0a_1)^{1/2}.
\]  

Because of the scaling \( E_{fas} \sim |a| \), the CBA wakefield can be considerably larger than that of the LWFA for \( a_0^2, a_1^2 \ll 1 \). As an example, we show the 1d-XOOPIC results for some model parameters in Figure 3. In this case, the two pulses collide at \( k_{pz} \approx 80 \), where one can see the approximate factor of 25 increase in the wake from \( k_{pz} > 80 \) (single pulse) to \( 15 < k_{pz} < 80 \) (pulse and pump). We find that to obtain the same wake using the LWFA, the laser power must be increased by more than an order of magnitude (from \( 2.7 \times 10^{16} \) W/cm\(^2\) to \( 10^{18} \) W/cm\(^2\)). This increase is less than the theoretically predicted factor of \( \sim 125 \) from (5). We present a more general LWFA-CBA comparison in Section 3.

3. COMPARISON OF CBA TO CONVENTIONAL LWFA

As we have seen in the previous section, one can use the counter-propagating pump of the colliding beam accelerator to significantly enhance the peak longitudinal field of a single pulse by 1-2 orders of magnitude. From the theoretical formulae given by (4) and (5), we predict that for small intensities \( a_1a_0 < (\omega_p/\omega_o) \), a pump of amplitude \( a_1 \) increases the LWFA wake by a factor of \( 16a_1^2(\omega_o/\omega_p)^5 \). As the pump and/or pulse become more intense, the wake amplification scales as \( 1/a \). Thus, the CBA appears to be an attractive alternative to the LWFA for interesting accelerator research at low power. Because the CBA still requires a short, resonant pulse of nontrivial amplitude (and, hence, a CPA system), it’s natural to ask is whether it would be easier or more cost effective to implement the counter-propagating, detuned geometry of the CBA at lower power, or to improve the amplifier and optics for high-power LWFA experiments. This question depends in part on technology requirements and capabilities that are constantly changing. Thus, we will simplify the problem by assuming an existing CPA producing a pulse of minimum width \( \sigma \) and a certain total energy budget, and remember that any findings must inevitably be weighted by experimental considerations.

To accelerate electrons over a length \( L \), the CBA requires a pump at least of length \( 2L \). For our case, we will consider the acceleration length to be equal to the dephasing length \( L = L_d = \lambda_p(\omega_o/\omega_p)^2 \). If one assumes the acceleration length to be the depletion length, our conclusions still hold, although one must also consider the relevant technological issues involved for each (the CBA requires multiple, differently detuned pumps, while a tapered plasma is proposed for the LWFA [12]). For this reason, we assume a CBA pump length of \( L_{pump} = 2L_d = 2\lambda_p(\omega_o/\omega_p)^2 \), and the total energy requirement is

\[
U_{CBA} \propto 2\sqrt{\pi}a_0^2 + 2\pi(\omega_o/\omega_p)^2a_1^2.
\]  

The resulting CBA wakefield has the limiting formula given by (4) for the linear case and (5) for the particle trapping case. On the other hand, the energy requirement for a
single resonant LWFA pulse of amplitude \( a_g \) is given by

\[
U_{LWFA} \propto 2\sqrt{\pi}a_g^2.
\]  

First, we consider the case of that the CBA wake is linear, and (4) holds. Equating the CBA and LWFA energies (6) and (7), and applying the conditions for wake enhancement \([4a_1 > (\omega_p/\omega_0)^{3/2}]\) and linearity \([\omega_B^2 = 4a_0a_1\omega_0^2 < \omega_p^2]\), one can show that the single pulse LWFA generates higher gradients than a CBA of identical total energy if \((\omega_p/\omega_0) > 10^{-3}\). For the linear analysis to be valid, these tenuous plasmas require very low intensity lasers, so that \(E_{fast} < (mc\omega_0/e) \times 10^{-4}\). As an example, for 1 \(\mu\)m light in the linear regime, this requirement implies that if the CBA produces a larger plasma response than the LWFA of identical total energy, the wakefield can be at most 1 MeV/m, and cannot compete with conventional devices. For CBA parameters identical to those of the simulation in Figure 2, a short pulse of identical energy would have \(a_g = 0.8\), and a nonlinear wake approximately 20 times that of the CBA.

In the particle trapping regime, setting (6) equal to (7) and using the particle trapping condition \(\omega_B^2 > \omega_p^2\), one can show that for a given laser energy

\[
E_{LWFA} > \left[\frac{1}{2\sqrt{e}}\left(\sqrt{\pi}a_0 + \frac{\pi}{a_0}\right)\right] E_{CBA}.
\]  

Since the term in square brackets of (8) is always greater than one, we arrive at an even stronger conclusion than in the linear case: for a fixed total energy and acceleration length \(L = L_d\), the single pulse LWFA always outperforms the CBA. As an example, using the energy available for the CBA parameters of Figure 3 would give rise to a nonlinear wake about 2.5 times that of the CBA with a pulse \(a_g = 1.33\).

To complete our discussion, we note that given the additional constraint of peak intensity (along with pulse width and energy) the single pulse LWFA is no longer the optimal solution. In this case, one can imagine using the available laser energy to inject a train of short pulses separated by integer values of the plasma period. In the linear regime \(a_g^2 \ll 1\), a sequence of \(N\) such pulses with the same intensity as the CBA seed will have the wake described in (2) and energy in (7), with \(a_g^2\) replaced by \(Na_0^2\). Because of this simple substitution, our previous comparison analysis still holds unchanged, but the additional technical difficulties apparent in such a carefully timed pulse train might limit its usefulness.

**APPENDIX: PHYSICAL ELECTRIC FIELDS IN PIC CODES**

When evaluating the role of pulse-shaping in optimizing LWFA performance, it is important to ensure that the field profiles considered are physically realizable, at least in principle, or else spuriously large wakefields can result. Typically in PIC codes, radiation fields are launched as boundary conditions on the electric field at one or more surfaces outside the plasma, often in the form of a sinusoidal carrier modulated by a prescribed envelope function. While these pulse shapes will satisfy the vacuum Maxwell equations, one must also take care that their corresponding “sources” be physically realizable.
In particular, we will see that under reasonable assumptions, the radiative field components must average to zero at any fixed observation point sufficiently far from the actual source of the radiation. Such limitations were discussed as far back as [13], and have more recently been invoked in the debate surrounding vacuum laser acceleration [14]. Radiation fields launched in the vacuum with a non-trivial DC component can excite unphysically large longitudinal fields or erroneous transverse fields after passing through the plasma. In one-dimensional XOOPIc simulations of short-pulse wakefield generation, inaccurately large wakefields were excited by electric fields with Gaussian envelopes whose widths were approximately ten laser wavelengths or less. Similarly, longer envelopes which rise or fall too steeply (such as step functions) can also yield spurious results.

In three dimensions, the usual argument proceeds by noting that physically realizable sources for laser fields will consist of a set of point charges which, although temporarily accelerated, remain bounded in space for all time; that is, at any time \( t \), all charges contributing to the radiation lie within some finite distance \( R \) from, say, their centroid \( \vec{x}_0 \) at \( t = 0 \).

If \( \vec{E}(\vec{x}, t) = -\frac{1}{c} \frac{d}{dt} \vec{A}(\vec{x}, t) \) is the radiative component of the electric field, such that \( \vec{A}(\vec{x}, t) \) is the vector potential in the Coulomb gauge (\( \nabla \cdot \vec{A}(\vec{x}, t) = 0 \)), then for any observation point \( \vec{r} \) sufficiently distant from all sources, i.e., \( |\vec{r} - \vec{x}_0| \gg R \), we must have

\[
\int_{-\infty}^{\infty} dt \vec{E}(\vec{r}, t) = \frac{1}{c} \vec{A}(\vec{r}, -\infty) - \frac{1}{c} \vec{A}(\vec{r}, +\infty) = 0. \tag{9}
\]

For supposing otherwise, in the frequency-domain representation of the field, namely

\[
\vec{E}(\vec{r}, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \vec{E}(\vec{r}, t), \tag{10}
\]

a non-vanishing DC component, i.e., \( \vec{E}(\vec{r}, \omega = 0) \neq 0 \) is present. But any DC component of the electric field will necessarily correspond to a static solution to Maxwell equations, so that far from the sources it must fall off as \( |\vec{E}(\vec{r}, \omega = 0)| \sim 1/|\vec{r} - \vec{x}_0|^2 \) (as in the Coulomb or Biot-Savart laws). It can easily be shown from energy conservation arguments, however, that true radiative fields must fall off with the slower \( 1/|\vec{r} - \vec{x}_0| \) scaling.

On the other hand, in truly one-dimensional geometries, sources may be interpreted as transverse charge sheets of uniform surface charge density, which are not bounded in three-dimensional space, and whose fields do not fall off with distance, so the above argument fails. Yet, in one-dimensional situations, it is particularly critical that electromagnetic fields average to zero to obtain physically realistic results, because transverse canonical momentum is exactly conserved. From (9), a non-vanishing DC component of the radiative electric field implies \( \vec{A}(\vec{r}, t = \infty) \neq \vec{A}(\vec{r}, t = -\infty) \). In order that transverse canonical momentum remain conserved in the presence of this unphysical field, a plasma electron oscillating around the position \( \vec{r} \) under the influence of the field will be left with a compensatory spurious component of kinetic momentum, which through pondermotive effects in the tail of the pulse can generate unphysical longitudinal wakefields, and also can act as a source for spurious transverse electromagnetic components.
In a PIC simulation of short pulses, this can have disastrous effects, as shown in Figure 4. Note that the error involved becomes quite pronounced for pulses whose envelopes change on the order of the fast $\omega_0^{-1}$ time-scale, and that this error is solely due to the inappropriately launched electric field.

To see why fields must still average to zero in one dimension, recall that in this case the solenoidal current density is just the locally-transverse component of the full current density, and the Fourier transform of the Coulomb-gauge vector potential satisfies the Helmholtz equation

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} \right) \tilde{A}(z, \omega) = \tilde{J}(z, \omega) - (\hat{z} \cdot \tilde{J}(z, \omega)) \hat{z}. \quad (11)$$

For a source consisting of a possibly large but finite number of charge sheets, each of finite surface charge density, and each assumed to remain bounded in $z$ for all time, the Fourier transform $\tilde{J}(z, \omega)$ of the source current density, regarded as a function of $\omega$ at any fixed position $z$, can possess no poles at $\omega = 0$; the only possible singular contribution to the DC component might be a term proportional to a Dirac delta function $\delta(\omega)$ corresponding to some constant current density at one or more longitudinal positions. Higher-order derivatives of delta functions cannot appear: a term proportional to $\frac{d^n}{d\omega^n} \delta(\omega)$ in the frequency domain would correspond in the time domain to a contribution in the transverse current density growing as $\sim t^n$ indefinitely, which is impossible because only a finite amount of surface charge density can accumulate at any longitudinal position, and any charge must move with a subluminal velocity.

FIGURE 4. The folly of not enforcing $\int E \, dt = 0$. The solid line correctly predicts $E_z$, whereas the dotted line is off by a factor of 5. Note that the inaccurate pulse is simply a gaussian that does not average to zero because of the poor choice in the carrier phase; there isn’t anything “pathological” in the profile.
By assumption, one can take \( A(z = -\infty, \omega = 0) = 0 \), and integrating the Helmholtz equation in \( z \) for \( \omega = 0 \), one finds that for any \( z \), as \( \omega \to 0 \), \( A(z, \omega) \) remains either finite or, possibly, contains a term proportional to \( \delta(\omega) \). It follows that

\[
\lim_{\omega \to 0} \bar{E}(z, \omega) = \frac{i}{c} \lim_{\omega \to 0} \omega A(z, \omega) = 0.
\]  

In any case, avoiding such spurious fields in PIC codes is not difficult. At the boundary in question, one should choose a prescribed analytic form for the transverse vector potential \( \tilde{A}(t) \) rather than the electric field \( \bar{E}(t) \), and then determine the latter via \( \bar{E} = -\frac{1}{c} \frac{\partial}{\partial t} \tilde{A}(t) \), retaining all terms, i.e., not only the usual eikonal contribution, namely the derivative of the carrier multiplied by the pulse envelope, but also the carrier multiplied by the derivative of the envelope.

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