Temporal and spectral structure of an FEL oscillator during start-up

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Abstract
A previously developed Green's function technique for analyzing the time evolution of low-gain FEL oscillators is applied to the study of the start-up of oscillators from random beam noise. These calculations describe the evolution of the total radiation power and the spectral power as a function of pass number. The theory suggests that an estimate of the linear gain of an oscillator during start-up, as well as the noise level on the electron beam, can be inferred from a measurement of the position of the spectral peak of the FEL output. Analytical results are obtained for electron beams much longer than the slippage distance. A computationally efficient method for the spectral analysis of finite pulse FEL oscillators is described.

1. Introduction and general formalism

Much of the theoretical and numerical investigations of free-electron laser oscillators dealt with linear and nonlinear aspects of spectral evolution of the oscillators driven by long electron bunches [1]. A lot of work was also done on studying the linear properties of FEL oscillators with finite bunches [2,3], without including the effect of the electron shot noise. Analytical and numerical work on a start-up from the noise of finite-bunch FEL oscillators was done by Sparangle et al. [5] using an expansion in cavity modes of an optical resonator. Optical pulse length of most existing FEL oscillators is much shorter than mirror separation in an optical cavity. Thus, many cavity modes have to be kept in the expansion. In this paper, a novel approach to the start-up of an FEL oscillator is developed, which takes into account finite bunch length, cavity detuning, and slippage. This method does not require an introduction of a physically irrelevant (in the limit of a short optical pulse) frequency separation, determined by intermirror spacing, and, hence, is more computationally efficient and physically transparent.

The theoretical approximations we make are a one-dimensional system, a cold beam and low gain. With these assumptions, the FEL equations can be written in integral form [4]

\[
\frac{\partial \tilde{a}(\tilde{\zeta}, \tilde{\sigma})}{\partial \tilde{\zeta}} = iJ(\tilde{\sigma} - \tilde{\zeta}) \int_{0}^{\tilde{\zeta}} d\tilde{\zeta}'(\tilde{\sigma} - \tilde{\zeta}') \alpha(\tilde{\zeta} - \tilde{\zeta} + \tilde{\zeta}') e^{\gamma(\tilde{\zeta}' - \tilde{\zeta})},
\]

where we use the dimensionless variables

\[
\tilde{\zeta} = \frac{2\gamma c}{N_{w} \lambda_{w}} \left( \frac{t - \frac{\tilde{\zeta}}{c} }{t_{0}} \right),
\]

\[
\tilde{\sigma} = \frac{\frac{\tilde{\sigma}}{2k_{\lambda}}}{N_{w} \lambda_{w}},
\]

\[
y_{0} = 2\pi N_{w} \left( 1 - \frac{k_{\sigma}}{2k_{\lambda} \gamma t_{0}} \right).
\]

The dimensionless current is

\[
J(\tilde{\sigma}) = \frac{4\pi a_{w}^{2} N_{w}^{2} \gamma^{2} I_{0} I_{e}}{\gamma^{2} \tilde{y}^{2} I_{\lambda}} \eta(\tilde{\zeta}) = j_{\sigma} \eta(\tilde{\zeta}),
\]

where \( \gamma \) is the beam energy, \( r_{b} \) the beam radius, \( I_{\lambda} = 17 \text{ kA} \), the Allen current, \( a_{w} \) the wigglar parameter, and \( \eta(\tilde{\zeta}) \) gives the time dependence of the current and is normalized so that the current is \( J(\tilde{\sigma}) = I_{\text{em}}(\tilde{\sigma}) \). The notation mostly follows Ref. [4].

The formalism of Ref. [4] can be extended to include the noise on the beam and thus be used to investigate the
spectral and temporal properties during start-up of the FEL. Assume that at \( \tilde{z} = 0 \), particle phases are distributed according to

\[
\theta_f(\tilde{z} = 0) = \theta_{0j} + \sum_{m=1}^{\infty} \delta \theta_m(\tilde{s}_0, n) \cos(m \theta_{0j} + \delta_m),
\]

(6)

where

\[
\theta_{0j} = \frac{2\pi j}{N},
\]

\( \tilde{s}_0 = \tilde{s}(z = 0) \), and the number of particles in a ponderomotive bucket \( N \) is assumed large. In the linear regime, only the first harmonic contributes to FEL amplification.

With noise taken into account, Eq. (1) becomes

\[
\frac{\partial a(\tilde{z}, \tilde{s})}{\partial \tilde{z}} = i J(\tilde{s} - \tilde{z}) \int_{0}^{s} d\tilde{z}'(\tilde{z} - \tilde{z}') a(\tilde{s} - \tilde{z} + \tilde{z}') e^{i\varphi(\omega - \tilde{z})}
\]

\[
+ \frac{1 + a_n^2}{8\pi^2 N a} J(\tilde{s} - \tilde{z}) \tilde{\theta}_1(\tilde{s} - \tilde{z}, n) e^{-i\alpha},
\]

(7)

where \( \tilde{\theta}_1 = \delta \tilde{\theta}_1 \exp(i\delta_1) \).

As in Ref. [4], the radiation gain in a pass is found by integrating Eq. (7) over the wiggler length (which is physically equivalent to summing the contributions of all particles within a slippage distance of a given radiation slice \( \tilde{s} \)). This yields

\[
\Delta a(s; n) = i \int_{0}^{s} d\tilde{z} \int_{0}^{s} d\tilde{z}' J(\tilde{s} - \tilde{z}') a(\tilde{s} - \tilde{z} + \tilde{z}'; n) e^{i\varphi(\omega - \tilde{z}')}
\]

\[
+ \frac{1 + a_n^2}{8\pi^2 N a} \int_{0}^{s} d\tilde{z} J(\tilde{s} - \tilde{z}) \tilde{\theta}_1(\tilde{s} - \tilde{z}, n) e^{-i\alpha}.
\]

(8)

We next Fourier transform Eq. (8) in \( \tilde{s} \), assuming that the fields do not change significantly from pass to pass (which we have already assumed anyway by suppressing the dependence of \( a(\tilde{s}, \tilde{z}) \) on \( \tilde{z} \), the distance along the wiggler). In doing this, we assumed that the optical pulse is much shorter than the length of the optical cavity. Therefore, we use a free-mode expansion for the radiation field. If this assumption is not satisfied, expansion in cavity modes has to be employed, as was done by Sprangle et al. [5].

This gives, after including cavity detuning \( \sigma = 2\Delta \omega/(N \Delta \omega) \) and quality factor \( Q \),

\[
\frac{\partial \hat{a}(\omega; n)}{\partial n} + \left( i\omega \sigma + \frac{1}{2Q} \right) \hat{a}(\omega; n) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \hat{J}(\omega - \omega') G(\omega', \omega) \hat{a}(\omega'; n)
\]

\[
+ \frac{1 + a_n^2}{8\pi^2 N a} U(\omega - \omega_0)
\]

\[
\times \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \hat{J}(\omega - \omega') \tilde{\theta}_1(\omega', n),
\]

(9)

where \( \hat{J}(\omega) \) is the Fourier transform of the beam current profile \( J(\tilde{s}) \),

\[
G(\omega, \omega') = i \int_{0}^{s} \int_{0}^{s} d\tilde{z} d\tilde{z}'(\tilde{z} - \tilde{z}') e^{i(\omega - \omega')\tilde{z}'} e^{i(\omega - \omega_0)(\tilde{z} - \tilde{z}')},
\]

(10)

and

\[
U(\omega - \omega_0) = \frac{e^{i(\omega - \omega_0)} - 1}{i(\omega - \omega_0)}.
\]

(11)

Note that \( U(\omega - \omega_0) \) is the spectrum of spontaneous electron emission from a finite length wiggler.

2. Infinite electron beam

For an infinitely long electron beam Eq. (9) can be simplified by noting that in this case \( \hat{J}(\omega) = 2\pi \delta(\omega) \delta(\omega) \):

\[
\frac{\partial \hat{a}(\omega; n)}{\partial n} = - \left( i\omega \sigma + \frac{1}{2Q} \right) \hat{a}(\omega; n) + g_0(\omega - \omega_0) \hat{a}(\omega; n)
\]

\[
+ \frac{1 + a_n^2}{8\pi^2 N a} U(\omega - \omega_0) \tilde{\theta}_1(\omega, n),
\]

(12)

where \( g_0(\omega - \omega_0) \) is the standard detuning function:

\[
g_0(\omega - \omega_0) = \frac{\omega}{\omega_0} \int_{0}^{s} d\tilde{z} \int_{0}^{s} d\tilde{z}'(\tilde{z} - \tilde{z}') e^{i(\omega - \omega_0)(\tilde{z} - \tilde{z}')}.
\]

(13)

Note that \( g_0(\omega - \omega_0) \) is usually interpreted as a function of particle detuning (given by \( \omega_0 \)) for a fixed radiation frequency \( \omega = 0 \). Here we consider \( g_0(\omega - \omega_0) \) as a spectral function for a fixed electron energy. Stochastic equation (12) can be exactly integrated if assumptions are made about the statistics of particle noise. Assuming that shot noise on successive electron bunches is not correlated and coarse-graining on a scale much longer than a ponderomotive wavelength (that is, on a scale \( \delta \tilde{s} \gg 1/N \)), one can show that

\[
\langle \tilde{\theta}_1(\tilde{s}, n) \tilde{\theta}_1(\tilde{s}, n) \rangle = \frac{N}{N_\Delta} \delta(\tilde{s}_1 - \tilde{s}_2) \delta(n_1 - n_2),
\]

\[
\langle \tilde{\theta}_1(\omega, n) \tilde{\theta}_1(\omega, n) \rangle = \frac{L^2/\Delta \omega^2}{N_\Delta} \delta(n_1 - n_2),
\]

(14)
where $l_b$ is the length of the electron beam. The average spectral power as function of pass number can then be evaluated as

$$|a(\omega, n)|^2 = \left(1 + \frac{a_w^2}{2} \right)^2 \frac{\sin^2(\omega - y_0/2 \lambda_b e^{2\pi \sigma} - 1)}{16\pi^4 N w^2 N_j \lambda_w} (\omega - y_0)^2 \frac{1}{\lambda_b 2\pi \sigma} (\omega - y_0).$$

(15)

where

$$g(\omega - y_0) = g_0(\omega - y_0) - \left(i\omega \sigma + \frac{1}{2Q}\right).$$

It follows from Eq. (15) that initially, for $n \leq 1/2g_{\sigma}(\omega - y_0)$, the radiation power grows linearly with pass number, and is centered at $\omega = y_0$. This is as expected, the power is first centered at the peak of spontaneous emission. For large pass numbers $n > 1/2g_{\sigma}(\omega - y_0)$, radiated power grows exponentially with $n$ and its spectrum is shifted in frequency to $\omega - y_0 \approx -2.6$. Inspection of Eq. (15) shows that for large $n$, only a small amount of the spontaneously radiated power, corresponding to the power radiated at frequencies close to $\omega - y_0 \approx -2.6$ contributes to the exponential growth. The power of the detuning near peak gain is reduced by a factor of 7.3 from the peak spectral density of spontaneous radiation at $\omega = y_0$, this factor is somewhat analogous to the launching losses in an FEL amplifier.

3. Finite size beams: numerical algorithm

For a finite size electron beam, Eq. (9) cannot be solved analytically, and one has to resort to numerical integration of a discretized matrix equation. Assume a periodicity window $S$ which is split into $K$ temporal bins. The size of a bin should be small enough to capture the variation of beam density profile or the slipage distance, whichever is smaller, and wide in comparison with the beam size. A discrete Fourier transform implies the presence of non-physical (aliased) electron bunches, separated by distance $S$. It is important to ensure that, in a detuned cavity, the losses are large enough (or $S$ is large enough) so that radiation does not have an unphysical coupling between adjacent aliased electron bunches. Of course, if the accelerator produces a stream of closely spaced bunches, the bunch separation will define the periodicity window. Splitting the frequency domain into $K$ bins separated by $\Delta \omega = 2\pi/S$, and assuming for notational convenience $\omega_k = k\Delta \omega$ (where $k$ is an integer number between 0 and $K - 1$), Eq. (9) is discretized as

$$\frac{\partial \hat{a}_k(n)}{\partial n} + \left(\omega_k \sigma + \frac{1}{2Q}\right) \hat{a}_k(n) = \sum_{j=0}^{K-1} \frac{\Delta \omega}{2\pi} J_{k-j} g_{kj} \hat{a}_j(n) + \frac{1}{8\pi^2 N w a_w} \sum_{m=0}^{K-1} \frac{\Delta \omega}{2\pi} J_{k-m} \hat{d}_{1m}(n),$$

(16)

where all indexed quantities are to be understood as their counterparts from Eq. (9), evaluated at the discrete frequencies

$$u_k = u(\omega = k\Delta \omega).$$

(17)

Note that in the limit of an infinite beam

$$\frac{\Delta \omega}{2\pi} J_{k-j} \to \delta_{kj},$$

where $\delta_{kj}$ is a Kronecker delta. Introducing

$$T_{kj} = \frac{\Delta \omega}{2\pi} J_{k-j} g_{kj} - \left(\omega_k \sigma + \frac{1}{2Q}\right) \delta_{kj}$$

and

$$N_{kj} = \frac{1}{8\pi^2 N w a_w} \frac{\Delta \omega}{2\pi} J_{k-j},$$

we rewrite Eq. (16) in matrix form

$$\frac{\partial \hat{a}}{\partial n} = T \hat{a} + N \hat{d}_1(n),$$

(18)

where $\hat{a}$ and $\hat{d}_1$ are complex contravariant vectors (columns) and $T$ and $N$ are complex square matrices.

The noise $\theta$ is the uncorrelated (from pass to pass) white noise, characterized by the statistics:

$$\langle \theta_1(n_1)\theta_1^*(n_2) \rangle = \frac{SN w}{N_j} \delta(n_1 - n_2) I,$$

where $I$ is a unit matrix, and $\dagger$ designates Hermitian conjugation (transposed and complex conjugated). The spectral properties of the output radiation are characterized by the spectral matrix

$$P_{kj} = \hat{d}(\omega = k\Delta \omega) \hat{d}^*(\omega = j\Delta \omega).$$

(19)

Symbolically,

$$P = \hat{a}\hat{a}^\dagger.$$

Diagonal elements of $P$ characterize the spectral power density of the output radiation, as a function of pass number. A closed-form expression for $P(n)$ can be obtained:

$$P(n) = e^{\sigma T} \int_0^n d\omega \left(e^{-\sigma T NN^* e^{-\sigma T}}\right) e^{\sigma T}.$$

(20)

IV. LONG-WAVELENGTH FELs
Eq. (20) can be greatly simplified using the results of Ref. [6]. If
\[
\exp \left( \begin{bmatrix} T^* & N N^* \\ 0 & -T^* \end{bmatrix} \right)^n = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix},
\]
(21)
then
\[
F_{12} F_{12} = \int_0^n \text{d}n' (e^{-n' T} N N^* e^{-n' T'}).
\]
(22)
Computing an exponent of an extended matrix in Eq. (21) can be done very efficiently by successive squaring of the matrix [4], thus establishing matrices \( F_{12} \) and \( F_{12} \). Matrices \( e^{nT} \) and \( e^{n'T'} \) are then computed in a similar fashion.

In conclusion, we developed a general one-dimensional theory of start-up of a low-gain FEL oscillator. This theory takes into account radiation slip-page, cavity losses and desynchronization, and arbitrary electron bunch shapes. A computationally efficient method for solving a finite-bunch problem was developed.

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References