Effect of Longitudinal Space-Charge Waves of a Helical Relativistic Electron Beam on the Cyclotron Maser Instability

C. Chen, B. G. Danly, Member, IEEE, G. Shvets, and J. S. Wurtele

Abstract—The influence of the longitudinal space-charge waves of a coherently gyrophased, helical relativistic electron beam on the cyclotron maser instability is investigated in a cylindrical waveguide configuration using a three-dimensional kinetic theory. A dispersion relation that includes waveguide effects is derived. The stability properties of the cyclotron maser interaction are examined in detail. It is shown that, in general, the effects of space-charge waves on a coherently gyrophased beam are suppressed in a waveguide geometry, in comparison with an ideal one-dimensional cyclotron maser with similar beam parameters.

I. INTRODUCTION

It is well-known that in ideal one-dimensional systems, the longitudinal electrostatic waves of an isotropically gyrophased relativistic electron beam decouple from the co-propagating transverse electromagnetic waves in a uniform magnetic field. The interaction of the fast electromagnetic wave and the beam cyclotron mode, driven by an inverted population in the perpendicular momentum of the electron distribution function \( f_0(p_x^2, p_z) \), leads to the cyclotron maser instability [1], [2]. This provides the physical mechanism for coherent radiation generation in relativistic cyclotron autoresonance masers [3]–[8] and gyrotrons [3], [9].

In the early 1970's, Kotsarenko et al. [10] realized that both space-charge waves and beam cyclotron modes can interact with the fast electromagnetic wave, provided that the initial distribution of the electron gyrophases is nonuniform or coherent. An example of such a coherently gyrophased beam is a helical beam that is often transformed from a solid beam by means of a wiggler or kicker magnet [6]–[8]. Recently, Fruchman and Friedland developed a one-dimensional linear theory using both a cold-fluid model [11] and a kinetic model [12]. Kho et al. [13] subsequently performed one-dimensional computer simulations and found good agreement with the kinetic theory. These analyses concluded that the longitudinal space-charge waves can strongly alter the stability properties of the cyclotron maser interaction. In particular, near the cyclotron resonance, the growth rate for a coherently gyrophased beam was found to be substantially larger than that for an isotropically gyrophased beam. Also, Antonsen et al. [14] investigated the effects of transverse space-charge waves in gyrotrons.

In this paper, a three-dimensional kinetic theory is developed describing the interaction of the longitudinal space-charge waves on a coherently gyrophased, helical relativistic electron beam with the beam cyclotron and transverse-electric (TE) waveguide modes in a cylindrical waveguide geometry. A dispersion relation that includes waveguide boundary conditions is derived. The detailed stability properties of the cyclotron maser interaction are examined. In comparison with the existing one-dimensional theory [12], [13], the results of this paper show that the presence of a waveguide plays an important role in suppressing longitudinal space-charge wave effects.

The basic physical mechanism for the excitation of longitudinal space-charge waves in the cyclotron maser (with \( k_\parallel \neq 0 \)) driven by a coherently gyrophased electron beam is similar to that in the free-electron laser [2], in the sense that there is a ponderomotive force bunching the electrons axially in configuration space. The ponderomotive force results from the beating between transverse electromagnetic perturbations and the coherent modulation of the electron transverse velocities induced by the wiggler magnetic field in the free-electron laser, or by the axial magnetic field in the cyclotron maser. However, in an ideal one-dimensional cyclotron maser, coherent velocity modulations occur only when the distribution of the electron gyrophases at the entrance of the interaction (\( z = 0 \)) is coherent; e.g., a delta function distribution where all particles have the same gyrophase.

The organization of this paper is as follows. After presenting the basic equations and assumptions in Section II, a dispersion relation is derived in Section III for the cyclotron maser interaction, including longitudinal space-charge waves and waveguide effects. The dispersion relation is analyzed in Section IV.

II. BASIC EQUATIONS AND ASSUMPTIONS

We consider a relativistic electron beam undergoing cyclotron motion in an applied uniform magnetic field \( B_0 \hat{e}_z \) and propagating axially through a lossless cylindrical waveguide of radius \( r_w \). The motion of each individual electron is described by the Lorentz force equation,

\[
\frac{d\vec{p}}{dt} = -e \left[ \vec{E} + \frac{\vec{v}}{c} \times (B_0 \hat{e}_z + \delta \vec{B}) \right]
\]

(1)

Manuscript received September 5, 1991; revised January 29, 1992. This work was supported by the Office of Basic Energy Sciences, the Division of High Energy Physics of the U.S. Department of Energy, and the Lawrence Livermore National Laboratory.

The authors are with the Plasma Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139.

IEEE Log Number 9108087.
and the electron phase-space density function \( f(\vec{x}, \vec{p}, t) \) evolves according to the Vlasov equation

\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} - e \left[ \vec{E} + \frac{\vec{v}}{c} \times (B_0 \vec{e}_z + \delta \vec{B}) \right] \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{2}
\]

In (1) and (2), \(-e\) is the electron charge, \(c\) is the speed of light in vacuo, and the perturbed electric and magnetic fields, \(\delta \vec{E}(\vec{x}, t)\) and \(\delta \vec{B}(\vec{x}, t)\), include electromagnetic and space-charge wave contributions.

### A. Equilibrium State

In the absence of perturbations (\(\delta \vec{E} = 0 \equiv \delta \vec{B}\)), the exact constants of motion for an individual electron in the applied magnetic field \(B_0 \vec{e}_z\) are the electron guiding center radius and angle, \(r_\theta\) and \(\theta_\phi\), the perpendicular and axial momentum components, \(p_{\perp} = (p_{\perp}^2 + p_{\theta}^2)^{1/2}\) and \(p_z\), and the quantity \(\phi - \Omega_p t / p_z\), where \(\Omega_p\) is the electron rest mass, \(\Omega_c = eB_0 / m_0c\) is the nonrelativistic cyclotron frequency, and \(\phi = \tan^{-1}(p_\theta / p_z)\) is the gyrophase of the electron (see Fig. 1). Therefore, the equilibrium distribution function describing a cold, coherently gyrophased relativistic electron beam can be expressed as

\[
f_0(r_\theta, \phi - \Omega_p t / p_z, p_{\perp}, p_z) = \frac{n_b}{p_{\perp, 0}} G(r_\theta) \delta(p_{\perp} - p_{\perp, 0}) \delta(p_z - p_{z, 0}) \delta \left( \phi - \frac{\Omega_p t}{p_z} \right) \tag{3}
\]

with \(\int G(r_\theta) r_\theta dr_\theta d\theta_\phi = 1\), \(r_\perp = [r^2 + r_L^2 + 2r_L \sin(\theta - \phi)]^{1/2}\), and \(r_L = p_{\perp, 0} / m_0\Omega_c\). In (3), \(n_b\) is the number of electrons per unit axial length, and \(\delta\) is the delta function. The electron density in cylindrical coordinates is given by

\[
n(r, \theta, z) = \int f_0(r_\theta, \phi - \Omega_p t / p_z, p_{\perp}, p_z) p_{\perp, 0} dp_{\perp} dp_z d\phi = n_b G(r_\theta(r, \theta, z)), \tag{4}
\]

where \(r_\theta(r, \theta, z) = [r^2 + r_L^2 + 2r_L \sin(\theta - \Omega_p t / p_z)]^{1/2}\), and \(r_{\perp, 0} = p_{\perp, 0} / m_0\Omega_c\) is the Larmor radius of the electron with \(p_{\perp, 0} = p_{\perp, 0, 0}\). The density \(n(r, \theta, z)\) has an axial periodicity of \(L = 2\pi r_{\perp, 0} / m_0 c = 2\pi \gamma v_z / c\). The remainder of this analysis assumes an on-axis helical electron beam with

\[
G(r_\theta) = \begin{cases} 
1/\pi r_{gm}^2, & \text{if } 0 \leq r_\theta < r_{gm} \ll r_w \\
0, & \text{if } r_\theta > r_{gm}.
\end{cases} \tag{5}
\]

### B. Field Perturbations

In principle, the periodic equilibrium state \(f_0\) supports electromagnetic and space-charge wave perturbations of a Floquet type. In the present analysis, however, we assume that only one Floquet component in the radiation field, namely, a transverse-electric (TE\(_{mn}\)) mode, can strongly couple to the electron beam and space-charge waves. The electric and magnetic fields of the (TE\(_{mn}\)) mode are [4], [5]

\[
\delta \vec{E}(\vec{x}, t) = \frac{1}{2} \delta E^{(w)}(\vec{x}, t) \vec{e}_z = \frac{1}{2} \delta E^{(w)} c^{(w)} J_0(k_{\perp} r_{\perp}) \\
\cdot \exp \left( i k_{\perp}^w z - \omega t \right) \vec{e}_z + \text{c.c.} \tag{11}
\]

where \(\vec{e}_z = \vec{e}_x \times \vec{e}_y\). The subscript \(w\) denotes the transverse components of the electromagnetic perturbation, \(\delta E_{mn}\) is the amplitude of the TE\(_{mn}\) mode, and \(\omega = 2\pi f\) is the (angular) frequency of the perturbation. The vacuum TE\(_{mn}\) eigenfunction is defined by

\[
\Psi_{mn}(r, \theta) = C_{mn} J_m(k_{wm} r) \exp(i m \theta) \tag{9}
\]

with \(k_{wm}^2 / C_{mn}^2 = \pi (\nu - m^2) J_m^2(\nu)\). Here, \(J_m(x)\) is the Bessel function of first kind of order \(m\), and \(\nu = k_{wm} r_{w}\) is the nth zero of \(J_m(x) = dJ_m(x) / dx\). The function \(\Psi_{mn}\) satisfies the eigenvalue equation

\[
\left( \nabla_x^2 + k_{wm}^2 \right) \Psi_{mn}(r, \theta) = 0 \tag{10}
\]

the boundary condition \(\partial \Psi_{mn}(r = r_w, \theta) / \partial r = 0\), and the normalization condition \(\int_{\pi} \Psi_{mn}^* \Psi_{mn} d\theta = 1\).

To further simplify the analysis, we assume that the space-charge wave is longitudinal and azimuthally symmetric (\(\partial / \partial \theta = 0\)). Under this assumption, the space-charge wave field is approximately electrostatic and given by

\[
\delta \vec{E}^{(w)}(\vec{x}, t) = \frac{1}{2} \delta E^{(w)} c^{(w)} J_0(k_{\perp} r_{\perp}) \\
\cdot \exp \left( i k_{\perp}^w z - \omega t \right) \vec{e}_z + \text{c.c.} \tag{11}
\]
[The magnetic and transverse electric fields associated with the space-charge wave are of order \( J_\parallel (k_{\perp}^{(sc)} r, \phi, \theta) \sim \delta_{\phi, \theta} \delta_{r, \theta} \). In (11), \( k_{\perp}^{(sc)} = \mu / \gamma_0 \) is the transverse wave number of the space-charge wave, \( J_\parallel (\mu) = 0 \), and \( C^{(sc)} = k_{\perp}^{(sc)} \mu / \gamma_0 \) is a normalization factor. The wave number of the space-charge wave, \( k_{\parallel}^{(sc)} \), remains to be determined (see (40)). Furthermore, the present analysis assumes that \( J_\parallel (k_{\perp}^{(sc)} r) \) in (11) is the dominant component in the multimode expansion [5] of the space-charge wave.

\[ \frac{\partial E_{\perp}}{\partial t} + \nabla^2 \frac{\omega^2}{c^2} \delta \tilde{B} = -\frac{4\pi}{c} \nabla \times \delta \tilde{J} \]  

(12)

and that the space-charge wave satisfies the continuity-Poisson equation

\[ \frac{\partial \delta E_{\perp}^{(sc)}}{\partial t} = -4\pi \delta J_\parallel. \]

(13)

In (12) and (13), the current density perturbation is

\[ \delta \tilde{J}(\vec{x}, t) = -e \int \bar{v} f_1(\vec{x}, \vec{p}, t) d\vec{p} \]

(14)

with \( f_1(\vec{x}, \vec{p}, t) = f(\vec{x}, \vec{p}, t) - f_0(\vec{r}_g, \phi, \theta) \). Substituting (6)–(8) and (14) into (12), multiplying the equation by \( \Psi_{mn}^* \), and integrating over the cross section of the waveguide yields

\[ \frac{\omega^2}{c^2} - k_{\parallel}^2 - k_{mn}^2 \delta E_{mn} = \frac{8\pi i c}{c^2 k_{mn}} \int d\vec{p} d\vec{r} \times \nabla \Psi_{mn}^* \]

\[ \cdot \bar{v} f_1 \exp \left[ -i (k_{\parallel} z - \omega t) \right] \]

(15)

for the TE_{mn} mode. Similarly, from (11), (13), and (14), we have

\[ \delta E_{\parallel}^{(sc)} = \frac{8\pi i c}{c^2} \int d\vec{p} d\sigma C^{(sc)}(r) \delta J_\parallel(\vec{k}_{\perp}^{(sc)} r) \]  

\[ v_\parallel f_1 \exp \left[ -i \left( \frac{c}{c^2} k_{\parallel} - \omega t \right) \right] \]

(16)

for the space-charge wave.

\[ \delta \tilde{B} = \delta E_{\perp}^{(sc)} + \delta E_{\parallel}^{(sc)} \]

(17)

In (17), the usual total time derivative is replaced by the total derivative with respect to the axial distance \( z \), because the system has a single frequency and it is the spatial evolution of the perturbation that is of interest. Furthermore, it is convenient to express (17) in terms of the guiding-center phase-space variables \((r_g, \theta_g, z, p_L, \phi, p_z)\), as illustrated in Fig. 1. Using

\[ \frac{\partial E_{\parallel}^{(sc)}}{\partial \phi} + \delta E_{\phi} \]

(18)

where \( \beta_\perp = v_\perp / c \) and \( \beta_z = v_z / c \). The partial derivatives with respect to \( p_L \) and \( p_z \) are defined in the independent guiding-center phase-space variables

\[ \frac{1}{f_0} \frac{\partial f_0}{\partial p_L} = \frac{\delta'(p_z - p_{z0})}{\delta'(p_L - p_{L0})} \]

(19)

and

\[ \frac{1}{f_0} \frac{\partial f_0}{\partial p_z} = \frac{\delta'(p_z - p_{z0})}{\delta'(p_L - p_{L0})} + \frac{m_0 \omega_c}{p_z^2} \frac{\omega}{\omega_c} \frac{\delta'(p_z - p_{z0})}{\delta'(p_L - p_{L0})}. \]

(20)

In (18), we have neglected the terms proportional to \( G'(r_g) \) and \( \delta f_0 / \delta \phi \). Inclusion of these terms would add terms of order \( (\omega - k_{\parallel} v_{\perp} - B_{\perp} / c)^{-1} \) to the dispersion relation (42) and would not change our basic conclusions.

\[ \frac{\partial}{\partial t} \left( \frac{\partial f_0}{\partial p_L} \right) = \delta'(p_z - p_{z0}) \]

(21)

\[ \frac{\partial}{\partial t} \left( \frac{\partial f_0}{\partial p_z} \right) = \delta'(p_z - p_{z0}) \]

(22)

\[ \delta B_{\parallel} = \frac{c k_{\parallel}}{\omega} \delta E_{\perp} \]

(23)

Here,

\[ \Delta_{mn}(z, \phi, \theta, t) = k_{\parallel} z - \omega t + \phi - (l - m) \theta \]

(24)

\[ \Delta_{l}(z, \phi, \theta, t) = k_{\parallel} z - \omega t + l \phi - l \theta \phi + l \pi / 2 \]

(25)

\[ X_{mn}(r_g, r_L) = J_{l-m}(k_{\parallel} r_g) J_{l}(k_{\parallel} r_L) \]

(26)

\[ X_{l}(r_g, r_L) = J_{l}(k_{\parallel} r_g) J_{l}(k_{\parallel} r_L) \]

(27)
Substituting (21)–(23) into (18) yields
\[
\frac{df_1}{dz} = \frac{e}{2v_z} k_{mn} C_{mn} \delta E_{mn} \left[ \sum_{l=-\infty}^{\infty} X_{mnlt} \exp(i\lambda_{ml}) \right]
\cdot \left[ \frac{k_{l} v_z}{\omega} \frac{\partial f_0}{\partial p_{\perp}} + \frac{k_{l} v_z \partial f_0}{\omega} \frac{\partial f_0}{\partial p_z} \right] + \frac{e}{2v_z}
\cdot C^{(sc)} \delta E^{(sc)} \left[ \sum_{l=-\infty}^{\infty} X_{l}^{(sc)} \exp(i\lambda_{l}^{(sc)}) \right] \frac{\partial f_{0}}{\partial p_z} + \text{c.c.}
\]  

(28)

Integrating (28) along the characteristics given by \( t(z') = t + (z' - z)/v_z \) and \( \phi(z') = \phi + (\Omega_{l}/\gamma)(z' - z)/v_z \), we find
\[
f_1 = f_{1}^{(sc)} + \text{c.c.}
\]  

(29)

and
\[
f_{1}^{(sc)}(r_{y}, \theta_{y}, z, p_{\perp}, p_z, \phi, t)
= \frac{e}{2\pi} \sum_{l=-\infty}^{\infty} \left\{ k_{mn} C_{mn} X_{mnlt} \delta E_{mn} \exp(i\lambda_{ml}) \frac{\beta_{l} X_{l}^{(sc)}}{k_{l} v_z - \omega + \Omega_{l}/\gamma} \right\} f_{0}.
\]  

(30)

The operators \( O, O_{l}^{(TE)}, \) and \( O_{l}^{(sc)} \) are defined by
\[
O = \left( 1 - \frac{k_{l} v_z}{\omega} \right) \frac{\partial}{\partial p_{\perp}} + \frac{k_{l} v_z}{\omega} \frac{\partial}{\partial p_z}
\]  

(31)

\[O_{l}^{(TE)} = O + \frac{i v_z}{k_{l} v_z - \omega + \Omega_{l}/\gamma} \frac{m_{O} \Omega_{l}}{p_{\perp}^{2}} \frac{\partial}{\partial \phi}
\]  

(32)

and
\[
O_{l}^{(sc)} = \frac{\partial}{\partial p_z} + \frac{i v_z}{k_{l} v_z - \omega + \Omega_{l}/\gamma} \frac{m_{O} \Omega_{l}}{p_{\perp}^{2}} \frac{\partial}{\partial \phi}
\]  

(33)

Substituting (30) into the wave equation (15), and expanding \( \Psi_{mn}(r, \theta) \) in terms of guiding-center variables [as used in (21)], we obtain
\[
D_{mn}^{(TE)}(\omega, k_{l}) \delta E_{mn} = \chi(\omega, k_{l}^{(sc)}) \delta E^{(sc)}
\]  

(34)

where
\[
D_{mn}^{(TE)}(\omega, k_{l}) = \frac{\omega^2}{c^2} - k_{l}^2 - k_{mn}^2 - 4\pi e^2 \left( \frac{\omega}{c} \right) C_{mn}^{2}
\cdot \sum_{l=-\infty}^{\infty} \int d\sigma d\phi \beta_{l} X_{l}^{(sc)} \frac{f_{0}}{k_{l} v_z - \omega + \Omega_{l}/\gamma}
\]  

(35)

\[
\chi(\omega, k_{l}^{(sc)}) = 4\pi e^2 \left( \frac{\omega}{c} \right) C_{mn}^{2} \frac{C^{(sc)}}{k_{mn}^{2}}
\cdot \sum_{l=-\infty}^{\infty} \int d\sigma d\phi \exp \left[ i \left( \Lambda_{l}^{(sc)} - \Lambda_{mn} \right) \right]
\cdot \frac{\beta_{l} X_{l}^{(sc)} \exp(i\lambda_{l}) f_{0}}{k_{l} v_z - \omega + (1 - m)\Omega_{l}/\gamma}
\]  

(36)

Similarly, combining (16) and (30), we have:
\[
D_{mn}^{l}(\omega, k_{l}) \delta E^{(sc)} = \kappa(\omega, k_{l}) \delta E_{mn}
\]  

(37)

where
\[
D_{mn}^{l}(\omega, k_{l}) = 1 - 4\pi e^2 \left( \frac{\omega}{c} \right) C_{mn}^{2}
\cdot \sum_{l=-\infty}^{\infty} \int d\sigma d\phi \beta_{l} \left( \frac{X_{l}^{(sc)}}{k_{l} v_z - \omega + (1 - m)\Omega_{l}/\gamma} \right) f_{0}
\]  

(38)

\[
\kappa(\omega, k_{l}) = 4\pi e^2 \left( \frac{\omega}{c} \right) k_{mn} C_{mn} C^{(sc)}
\cdot \sum_{l=-\infty}^{\infty} \int d\sigma d\phi \exp \left[ i \left( \Lambda_{mn} - \Lambda_{l}^{(sc)} \right) \right]
\cdot \beta_{l} X_{l}^{(sc)} \frac{f_{0}}{k_{l} v_z - \omega + (1 - m)\Omega_{l}/\gamma}
\]  

(39)

To determine the coupling of the TE_{mn} mode and the space-charge wave, we need to evaluate the overlap integrals in (36) and (39). After a careful inspection of the integrals, we find that the relation,
\[
k_{l}^{(sc)} = k_{l} + m \frac{m_{O} \Omega_{l}}{p_{\perp}} \equiv k_{l} + m \frac{\Omega_{l}}{\gamma v_{z}}, \quad m = 1, 2, 3, \ldots
\]  

(40)

must be satisfied for the coupling. From (35) and (38), we obtain the dispersion relation,
\[
D_{mn}^{(TE)}(\omega, k_{l}) D_{mn}^{(sc)}(\omega, k_{l}) = \chi(\omega, k_{l}) \kappa(\omega, k_{l})
\]  

(41)

with \( k_{l}^{(sc)} \) defined in (40). This dispersion relation can be used to examine the stability properties of the cyclotron maser interaction of a fundamental or harmonic beam cyclotron mode with longitudinal, azimuthally symmetric space-charge waves and various TE modes.

In particular, for the cold, coherently gyrophased electron beam described in (3) and (5), to leading order in \((\omega - k_{l} v_{z} - (\Omega_{l}/\gamma)^{2})\), the dispersion relation (41) for \( l = m \) becomes
\[
D_{mn}^{(TE)}(\omega k_{l}) D_{mn}^{(sc)}(\omega, k_{l}) = \epsilon_{mn} f_{m}^{(sc)} \frac{\beta_{l} \omega - c k_{l}^{2}}{\Delta_{l}^{2}}
\]  

(42)

The dielectric functions for the TE_{mn} mode and the longitudinal space-charge wave, \( D_{mn}^{(TE)}(\omega, k_{l}) \) and \( D_{mn}^{(sc)}(\omega, k_{l}) \), are given by
\[
D_{mn}^{(TE)}(\omega, k_{l}) = \frac{\omega^2}{c^2} - k_{l}^2 - k_{mn}^2 + \epsilon_{mn} k_{mn} \frac{\omega^2 - c^2 k_{l}^2}{\Delta_{l}^2}
\]  

(43)

and
\[
D_{mn}^{(sc)}(\omega, k_{l}) = 1 - \frac{\epsilon_{l-m}^{(sc)}}{\Delta_{l}^{2}} \frac{(c k_{l}^{(sc)})^2}{\Delta_{l}^{2}}
\]  

(44)

respectively. Moreover, \( \Delta_{l} = \omega - k_{l} v_{z} - (\Omega_{l}/\gamma) \gamma_{l} = \left[ 1 + \frac{p_{\perp}^{2} + p_{\perp 0}^2}{m_{O} c^2} \right]^{1/2} - \frac{1}{2} \frac{\beta_{l}}{v_{z}/c} = p_{\perp} 0/\gamma_{m} m_{O} c \). Finally, the
dimensionless coupling constants $\epsilon_{\text{mnl}}$ and $\epsilon_{l-m}^{(sc)}$ are defined by

$$\epsilon_{\text{mnl}} = \frac{4\beta_2^2}{\gamma\beta_2} \left( \frac{I_b}{I_A} \right) \left[ J_{l-m}(k_{mn} r_m) J_l^*(k_{mn} r_L) \right]^2 \left( \frac{\nu_{mn} - m^2}{\nu_{mn}} \right) J_m^2(k_{mn}), \quad \text{(45)}$$

and

$$\epsilon_{l-m}^{(sc)} = \frac{4}{\gamma \beta_2} \left( \frac{I_b}{I_A} \right) \left[ \frac{J_{l-m}(k_{l-m}^{(sc)} r_m) J_{l-m}^*(k_{l-m}^{(sc)} r_L)}{k_{l-m}^{(sc)} \nu_{l-m} J_0^2(k_{l-m}^{(sc)} r_w)} \right]^2.$$

respectively. In (45) and (46), $I_b = e a_0 \beta_2 c$ is the beam current, and $I_A = m_0 v_0^2 / 2 \equiv 17$ kA, the Alfvén current. The dispersion relation (42) is a six-order polynomial of $k_{||}$, and therefore has six solutions for $k_{||}$. Each solution with a negative imaginary part corresponds to an unstable mode with a spatial growth rate of $-\text{Im} k_{||} > 0$.

Three remarks on the dispersion relation (42) are in order. First, the coupling between the longitudinal space-charge wave and the $TE_{mn}$ mode is proportional to $(\beta_2 \omega - ck_{||})^2$, which is in agreement with the cold-fluid treatment of Kotsarenko et al. [10]. This quantity can be negligibly small in waveguide configurations. In fact, at grazing incident $\beta_2 \approx \beta_\phi \equiv c k_{||}/\omega$, it almost vanishes and therefore space-charge-wave effects are expected to be negligible. In any case, the influence of longitudinal space-charge waves is suppressed due to $k_{mn} \neq 0$ and $\beta_0 > 1$ in a waveguide geometry. On the contrary, in an ideal one-dimensional system in free space, because $\omega \cong ck_{||} = ck_0$ for electromagnetic waves propagating in the $z$ direction, $(\beta_2 \omega - ck_0)^2 \cong (1 - \beta_2^2) \omega^2$. The space-charge waves can then strongly modify the cyclotron maser interaction. [Note that in waveguide configurations, $(\beta_2 \omega - ck_{||})^2$ is less than $(1 - \beta_2^2) \omega^2$ at the upper intersection of the waveguide mode and beam cyclotron mode.]

Second, the three-dimensional dispersion relation (42) recovers, to leading order in $(\omega - k_{||} v_x - \text{Im} \Omega_{c1}/\gamma)^2$, the corresponding one-dimensional dispersion relation [12], [13] with the following substitutions:

$$l \rightarrow 1, \quad k_{||} \rightarrow k, \quad k_{mn} \rightarrow 0, \quad \epsilon_{l-m}^{(sc)}(ck_{l-m}^{(sc)})^2 \rightarrow \frac{\omega_{pb}^2}{\gamma}, \quad \epsilon_{l-m}(ck_{mn})^2 \rightarrow \frac{\omega_{pb}^2}{\gamma} \left( \frac{\beta_2^2}{2} \right). \quad \text{(47)}$$

Here, $\omega_{pb} = (4\pi e^2 n_c / m_0)^{1/2}$ is the nonrelativistic plasma frequency associated with a constant electron density of $n_c$ in the one-dimensional theory.

Third, in the case of an isotropically gyrophasor electron beam with an unperturbed distribution function depending on $p_\perp$ and $p_z$ but not on $\phi = m_0 \Omega_{c1} / p_z$, it can be shown that the space-charge wave decouples from the $TE_{mn}$ mode with $m \neq 0$, that the dispersion relation (42) reduces to

$$D_{mn}^{TE}(\omega, k_{||}) = 0, \quad m, n = 1, 2, 3, \ldots \quad \text{(48)}$$

for the $TE_{mn}$ mode, and

$$D^2(\omega, k_{||}) = 0 \quad \text{(49)}$$

for the space-charge wave. However, the space-charge wave does couple to the $TE_{mn}$ mode due to cold plasma oscillations.

IV. NUMERICAL RESULTS

In this section, the dispersion relation (42) is analyzed to illustrate the effect of the longitudinal space-charge wave on the cyclotron maser interaction in a cylindrical waveguide configuration for the coherently gyrophasor, helical relativistic electron beam described in (3) and (5). The results are compared with the corresponding isogyrophasor beam, as well as with the corresponding coherently gyrophasor beam in an ideal one-dimensional system.

Fig. 2 shows the gain bandwidth of the $TE_{11}$ mode for the following choice of system parameters: beam current $I_b = 500$ A, beam energy $\gamma = 4.0$, normalized perpendicular velocity $\beta_1 = 0.36$, $l = 1$, axial magnetic field $B_0 = 18.94$ kG, waveguide radius $r_w = 0.3$ cm, $r_{pm} \approx 0$, $k_{l-m}^{(sc)} = \mu / r_w = 2.405 / r_w$, and $k_{||} = v_{l1} / r_w = 1.8412 / r_w$.

![Fig. 2](image)

for the space-charge wave. However, the space-charge wave does couple to the $TE_{mn}$ mode due to cold plasma oscillations.

Several interesting features are illustrated in Fig. 2. First, it is evident that inclusion of the space-charge wave broadens the gain bandwidth of the fast unstable $TE_{11}$ mode (corresponding to the dashed curve in the frequency range from $\omega / ck_{11} \cong 1.2$ to 42). Similar behavior has been found in ideal one-dimensional systems [12], [13]. Second, as discussed in Section III, the influence of the space-charge wave on the cyclotron maser interaction is small, particularly near the cyclotron resonant frequencies $\omega / ck_{11} \cong 3.24$ (or $\omega / 2\pi \cong 95$ GHz) and $\omega / ck_{11} \cong 1.5$ (or $\omega / 2\pi \cong 44$ GHz). For example, at $\omega / 2\pi = 95$ GHz, the growth rate for the
Fig. 3. The solid and dashed curves show the gain bandwidth for the cyclotron maser interaction at grazing incidence for coherently and isotropically gyrophase electron beams, respectively. The choice of system parameters corresponds to the NRL gyro-TWT amplifier experiment: $I_0 = 500$ A, $\gamma = 2.76$, $\beta_0 = 0.4$, $l = 1$, $B_0 = 8.0$ kG, $r_w = 0.54$ cm, $r_{wm} \simeq 0$, $k_{\parallel}(e) = \mu_e r_w = 2.405/r_{wm}$, and $k_{\perp} = \nu_{\perp} r_w = 1.8412/r_{wm}$. The operating frequency is 35 GHz or $\omega/c k_{11} = 2.01$.

coherently gyrophase beam is only 3% larger than that for the isotropically gyrophase beam in the waveguide configuration. In contrast, the growth rate for the coherently gyrophase beam is found to be 60% larger than that for the isotropically gyrophase beam using the one-dimensional theory [12], [13] and the corresponding $\omega_p^2/c^2 = 4 I_0/3 \beta_0 r_L^2 I_A = 7.7 ~ \text{rad}^2/\text{cm}^2$. Third, space-charge wave effects cause instability in the frequency range from $\omega/c k_{11} \equiv 4.2$ to 4.5, (corresponding to $\beta_0 \equiv 1.03$), where the net axial and inertial bunching vanish in the case of the isotropically gyrophase beam [5], [16].

Fig. 3 shows the gain bandwidth for a coherently and an isotropically gyrophase electron beam corresponding to the gyro-TWT amplifier experiment conducted at the Naval Research Laboratory (NRL) [7]. The system parameters are $I_0 = 500$ A, $\gamma = 2.76$, $\beta_0 = 0.4$, $l = 1$, $B_0 = 8.0$ kG, $r_w = 0.54$ cm, $r_{wm} \simeq 0$, $k_{\parallel}(e) = \mu_e r_w = 2.405/r_{wm}$, and $k_{\perp} = \nu_{\perp} r_w = 1.8412/r_{wm}$. The amplifier operated near grazing incidence at $\omega/2\pi = 35$ GHz (or $\omega/c k_{11} = 2.01$). Although the bandwidth (solid curve) for the coherently gyrophase beam is wider than the bandwidth (dashed curve) for the isotropically gyrophase beam, there is little difference at the operating frequency $\omega/c k_{11} = 2.01$ where $\beta_0 \omega - c k \parallel \simeq 0$ due to grazing incidence. This calculation suggests that the discrepancy [7] between the measured and theoretically predicted growth rates cannot be resolved by the inclusion of longitudinal space-charge into the theoretical analysis.

V. CONCLUSIONS

In conclusion, a three-dimensional kinetic theory in a waveguide geometry was developed for the cyclotron maser interaction, including the longitudinal, azimuthally phase space-charge waves of a coherently gyrophase, helical relativistic electron beam. A dispersion relation was derived and the detailed stability properties of the cyclotron maser interaction were examined. The results were compared with the existing one-dimensional theory [12], [13]. It was shown that, in general, the influence of longitudinal space-charge waves on a coherently gyrophase beam is suppressed in a waveguide geometry in comparison with an ideal one-dimensional cyclotron maser with similar beam parameters. A moderate enhancement of the growth rate was found near the cyclotron resonance for relatively high-current, high-density beams. Moreover, longitudinal space-charge wave effects were shown to be negligibly small near grazing incidence.

ACKNOWLEDGMENT

The authors wish to thank J. Davies for stimulating discussions and assistance in the analysis of the dispersion relation. The authors are also grateful to M. I. Petelin for discussions and for pointing out the work by Kotsarenko et al.

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