Particle Beam Stability in the Hollow Plasma Channel Wake Field Accelerator

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Abstract. The electromagnetic wake field response of a hollow plasma channel to a driver (laser or charged particle beam) of arbitrary velocity is derived. The dispersion and loss factors of excited fundamental and higher order azimuthal modes are computed. Growth rates for beam breakup instabilities are calculated and beam transport is studied. External focusing is shown to provide a method of controlling transverse instabilities. For parameters of interest for high gradient plasma-based accelerators, it is shown that the most severe limitation to the interaction length of a single accelerator stage based on the hollow plasma channel structure is the transverse instability of the particle beam.

INTRODUCTION

Plasma-based accelerators have the ability to sustain extremely large accelerating gradients, with possible high-energy physics applications [1,2]. Diffraction is a severe limitation for laser-driven plasma-based accelerators. Therefore, any successful design of a plasma-based accelerator which utilizes a laser driver must include some form of optical guiding. Two schemes for optical guiding are currently being explored for overcoming diffraction: relativistic self-focusing [3] and plasma channel guiding [4].

Relativistic self-focusing relies on the energy dependence of the plasma frequency to modify the plasma index of refraction. Relativistic self-focusing of long laser pulses (i.e., pulse lengths much longer than the plasma wavelength) suffer from Raman forward and sidescatter instabilities [5]. These instabilities lead to break up of the pulse into small pulses of order the plasma wavelength and therefore limit the propagation distance of the laser pulse. For short laser pulses (i.e., pulse lengths of order the plasma wavelength), such as those used in the standard laser wake field accelerator, relativistic self-focusing is substantially reduced. This is due to the generation of a plasma density perturbation by the ponderomotive force of
the laser. For short pulses, the plasma frequency decrease from relativistic effects is balanced by this density perturbation [4]. Consequently, the index of refraction will have no transverse variation, and the plasma cannot optically guide a short laser pulse.

Plasma channel guiding provides optical guiding through the use of a plasma channel that has a higher plasma density outside the channel than inside the channel giving the plasma channel an index of refraction which decreases from the channel axis. A fixed plasma channel is analogous to an optical fiber and its guiding properties can be similarly analyzed. Plasma channels can be used to guide short pulses and have been studied analytically using axisymmetric models for a parabolic plasma density variation [4] and for hollow plasma channels [6].

Calculations show that a hollow plasma channel, in addition to optically guiding the laser pulse, supports a plasma wave with attractive properties for particle acceleration. The driver excites a surface mode in the plasma which extends into the channel. Unlike in a homogeneous plasma or parabolic channel, the transverse profile of the driver is decoupled from the transverse profile of the accelerating mode. Therefore, for a relativistic driver, the accelerating gradient of the fundamental mode is uniform and the focusing fields are linear [6]. In addition, the excited fields in a hollow plasma channel are fully electromagnetic, unlike the electrostatic fields excited in a homogeneous plasma. These properties make a hollow plasma channel well-suited as a structure for both particle beam and laser-driven wake field accelerators.

Since the original demonstration of the guiding of a low-intensity laser pulse in a plasma channel at the University of Maryland [7], several research groups have been examining experimental methods of plasma channel formation and guiding of high-intensity lasers [8–11]. Methods of forming a plasma channel include: inverse bremsstrahlung heating of the plasma by a precursor laser pulse resulting in hydrodynamic expansion and channel formation [8,9] and discharge ionization of a preformed capillary tube [10,11].

In this paper an externally formed hollow plasma channel is characterized as an accelerating structure, independent of the structure excitation mechanism (laser or particle beam). The results provide the basic scalings for the plasma channel accelerator, including current limiting higher-order mode couplings. Higher-order moments of the drive pulse distribution, present due to drive pulse shape or misalignment, will excite higher-order modes in addition to the fundamental (accelerating) mode. These higher-order modes can cause beam instabilities, limiting the propagation distance and, therefore, the energy gain. In this paper we examine instabilities resulting from the beam-plasma coupling.
MODE STRUCTURE OF THE HOLLOW PLASMA CHANNEL

Our derivation of the excited fields in the hollow plasma channel is similar to that used for conventional metallic structures. The driver (beam current or laser pulse) is assumed to be unaltered during the excitation of the channel. The hollow plasma channel is modeled as having an equilibrium electron plasma density $n_e(r) = n_0 \Theta(r - r_w)$, where $\Theta$ is the Heaviside step function, $r_w$ is the radius of the channel wall, and $n_0$ is the number density of the plasma outside the channel. The ion plasma density is assumed to be equal to the equilibrium electron plasma density. The ions are also assumed to remain motionless since the drive pulse duration is taken to be much shorter than the response time of the ions.

For a hollow plasma channel, $\nabla n_e = 0$ in all regions, and the wave equation for the electric field $\vec{E}$, obtained from the Maxwell equations and the linearized cold collisionless fluid equations modeling the plasma response, is

$$\left(c^2 \nabla^2 - \frac{\partial^2}{\partial t^2} - \omega_p^2\right) \vec{E} = 4\pi \frac{\partial}{\partial t} \vec{J}_{\text{ext}} + 4\pi c^2 \nabla \rho_{\text{ext}},$$

where $\omega_p$ is the plasma frequency. Here $\vec{J}_{\text{ext}}$ and $\rho_{\text{ext}}$ are the external drive current and charge densities respectively. Note that if $\nabla n_e$ is nonvanishing, then resonant absorption is possible in the plasma channel walls and the excited fields can mode convert into an electrostatic Langmuir wave. The electromagnetic fields in the channel decay as the electrostatic Langmuir wave in the plasma wall is excited. This leads to an effective quality factor of this hollow plasma structure [12] and a corresponding limit on the number of bunches that can be accelerated.

The source terms in the wave equation are determined by the external driver. For beam-driven excitation of the plasma channel, $\vec{J}_{\text{ext}} = \vec{v}_b \rho_b$, where $\rho_b$ is the beam charge density and $\vec{v}_b$ is the beam velocity. For excitation by a laser pulse, the current source is driven, to lowest-order, by the ponderomotive force of the laser pulse envelope (i.e., the gradient of the radiation pressure). To second-order in $a < 1$, the current source driven by the ponderomotive force is

$$4\pi \frac{\partial}{\partial t} \vec{J}_{\text{ext}} = \frac{m_e c^2}{e} \omega_p^2 \nabla \frac{a^2}{2},$$

where $a = e|\vec{A}|/m_e c^2$ is the normalized vector potential of the laser.

For the linear analysis presented in this paper to be valid, surface plasma density perturbations should be small compared to the channel radius. This implies a laser pulse driver must satisfy $a^2 \ll 1$ (assuming the laser spot size is of order the channel radius $r_0 \sim r_w$ and the laser pulse duration is of order the plasma period $\omega_p T_L \sim 1$), and a particle drive beam must satisfy $N_b \ll (\omega_p/c) r_w^2 / r_e$, where $r_e = e^2/(m_e c^2)$ is the classical electron radius and $N_b$ is the number of electrons per bunch. For example, if $r_w = 20 \mu m$ and $n_0 = 7 \times 10^{16}$ cm$^{-3}$, then the linear theory is valid for beams with $N_b \ll 7 \times 10^9$ electrons.
The nonevolving driver propagates axially with group velocity near the speed of light \( c\beta \simeq c \). Thus, we make the “frozen-field” approximation: the modes are functions only of the co-moving coordinate \( \tau = t - z/(\beta c) \). The fields are decomposed into discrete azimuthal modes with mode index \( m \) and a Fourier transform in the co-moving coordinate \( \tau \) is performed such that solutions are of the form \( \exp(-i\omega_m \tau + im\theta) \), with the mode frequencies \( \omega_m \). The boundary conditions across the channel wall are: continuity of the electric and magnetic field components \( \epsilon_m \vec{E} \cdot \hat{r}, \vec{E} \times \hat{r}, \text{ and } \vec{B} \), where \( \epsilon_m = 1 - \omega_p^2(r)/\omega_m^2 \) is the dielectric function of the hollow plasma structure.

To study the excited channel modes synchronous with the driver, let \( \vec{E} = \hat{A}_r \epsilon_m(r, \theta) \exp(-i\omega_m \tau) \) and \( \vec{B} = \hat{A}_m b_m(r, \theta) \exp(-i\omega_m \tau) \), where \( \hat{A}_m \) are constants determined by the excitation mechanism. With these definitions, the equation for the plasma wave electric field behind the drive pulse is

\[
\left[ c^2 \nabla^2 - \omega_m^2 (\beta^{-2} - \epsilon_m) \right] \vec{E}_m = 0,
\]

where \( \nabla \perp \) is the transverse Laplacian. Note that only purely electromagnetic modes (i.e., \( \nabla \cdot \vec{E} = 0 \)) exist in the channel, and since there are no linear surface currents, the continuity of \( \nabla \times \vec{E} \) requires that the mode in the plasma also satisfies \( \nabla \cdot \vec{E} = 0 \).

For the fundamental mode \( m = 0 \), the solutions to the homogeneous wave equation Eq. (3) are

\[
e_{0x} = \frac{ik_1 c\beta}{\omega_0} \frac{I_0(k_1 r)}{I_1(k_1 r_w)},
\]

\[
e_{0r} = \frac{I_1(k_1 r)}{I_1(k_1 r_w)},
\]

\[
b_{0\theta} = \frac{\beta}{I_1(k_1 r_w)}
\]

in the channel \( r < r_w \), and

\[
e_{0x} = -\frac{ik_2 c\beta}{\omega_0} \frac{K_0(k_2 r)}{K_1(k_2 r_w)},
\]

\[
e_{0r} = \frac{1}{\epsilon_0} \frac{K_1(k_2 r)}{K_1(k_2 r_w)},
\]

\[
b_{0\theta} = \frac{\beta}{K_1(k_2 r_w)}
\]

in the plasma \( r > r_w \), where \( I_m \) and \( K_m \) are \( m \)-th-order modified Bessel functions of the second kind. Here \( k_1 = (\omega_m/c)(\beta^{-2} - 1)^{1/2} \) and \( k_2 = (\omega_m/c)(\beta^{-2} - 1 + \omega_p^2/\omega_m^2)^{1/2} \).

Note that in the limit of an ultra-relativistic driver (\( \beta \to 1 \)), \( k_1 \simeq 0 \) and \( k_2 \simeq \omega_p/c \).

The fundamental mode \( m = 0 \) frequency (eigenvalue equation) is

\[
\omega_0 = \omega_p \Omega_0 = \omega_p \left[ 1 + \frac{k_2 I_1(k_1 r_w) K_0(k_2 r_w)}{k_1 I_0(k_1 r_w) K_1(k_2 r_w)} \right]^{-1/2},
\]
where \( \Omega_m = \omega_m/\omega_p \) is the normalized frequency of the \( m \)th mode.

The higher-order modes \( m > 0 \) of the excited plasma wave are

\[
e_{mz} = \frac{k_1 c \beta}{\omega_m} \frac{I_m(k_1 r)}{I_m(k_1 r_w)} f(m\theta), \text{ and} \tag{11}
\]

\[
b_{mz} = \frac{k_1 c \beta}{\omega_m} \frac{\gamma}{I_m(k_1 r_w)} g(m\theta) \tag{12}
\]
in the channel \( r < r_w \), and

\[
e_{mz} = \frac{k_2 c \beta}{\omega_m} \frac{K_m(k_2 r)}{K_m(k_2 r_w)} f(m\theta), \text{ and} \tag{13}
\]

\[
b_{mz} = \frac{k_2 c \beta}{\omega_m} \frac{\gamma K_m(k_2 r)}{K_m(k_2 r_w)} g(m\theta) \tag{14}
\]
in the plasma \( r > r_w \), where \( f(m\theta) = \cos(m\theta) \) and \( g(m\theta) = -\sin(m\theta) \) for even modes or \( f(m\theta) = \sin(m\theta) \) and \( g(m\theta) = \cos(m\theta) \) for odd modes. In Eqs. (12) and (14), \( \gamma \) is

\[
\gamma = \frac{2m}{\beta} \left( \frac{k_1^2 - k_2^2}{r_w^2 k_1^2 k_2^2} \right) \frac{1}{\lambda} \left[ \frac{I_{m+1}(k_1 r_w) + I_{m-1}(k_1 r_w)}{k_1 r_w I_m(k_1 r_w)} + \frac{k_{m+1}(k_2 r_w) + k_{m-1}(k_2 r_w)}{k_2 r_w K_m(k_2 r_w)} \right]^{-1} \tag{15}
\]

The transverse fields for the higher-order modes can be computed directly from the axial components Eqs. (11)-(14) using the relations

\[
\bar{e}_{m\perp} = \frac{i c \beta^2}{\omega_m (1 - e_m \beta^2)} \left[ \mathbf{\hat{z}} \times \nabla \perp b_{mz} - \beta^{-1} \nabla \perp e_{mz} \right] \tag{16}
\]

\[
\bar{\mathbf{b}}_{m\perp} = \frac{-i c \beta^2}{\omega_m (1 - e_m \beta^2)} \left[ e_m \mathbf{\hat{z}} \times \nabla \perp e_{mz} + \beta^{-1} \nabla \perp b_{mz} \right] \tag{17}
\]

The eigenvalue equation for the higher-order modes is

\[
\frac{4m^2}{\beta^2 \left( \frac{k_1^2}{r_w^2} \frac{k_1^2 k_2^2}{k_2^2} \right)^2} \left[ \frac{I_{m+1}(k_1 r_w) + I_{m-1}(k_1 r_w)}{k_1 r_w I_m(k_1 r_w)} + \frac{k_{m+1}(k_2 r_w) + k_{m-1}(k_2 r_w)}{k_2 r_w K_m(k_2 r_w)} \right] \times
\]

\[
\left\{ \frac{I_{m+1}(k_1 r_w) + I_{m-1}(k_1 r_w)}{k_1 r_w I_m(k_1 r_w)} + \frac{e_m [K_{m+1}(k_2 r_w) + K_{m-1}(k_2 r_w)]}{k_2 r_w K_m(k_2 r_w)} \right\} = \frac{4m^2}{\beta^2} \left( \frac{k_1^2}{r_w^2} \frac{k_1^2 k_2^2}{k_2^2} \right)^2 \tag{18}
\]

The solutions of Eq. (18) provide the higher-order mode frequencies \( \omega_m \).
Ultra-Relativistic Limit

In the ultra-relativistic limit ($\beta \to 1$), the linearly excited mode frequencies of the hollow plasma channel (Eqs. (10) and (18)) become [13]

$$\omega_m = \omega_p \Omega_m = \omega_p \left[ \frac{(1 + \delta_{m0})(m + 1)K_{m+1}(R)}{2(m + 1)K_{m+1}(R) + RK_m(R)} \right]^{1/2},$$

(19)

where $R = \omega_pr_w/c$ is the normalized channel radius and $\delta_{m0} = 1$ for $m = 0$ and zero otherwise.

The forces on a beam due to the excited fields have attractive properties for particle acceleration. The excited fundamental mode fields in the channel (Eqs. (4)-(6)) provide the axial and transverse forces

$$F_z \simeq -eA_0 \cos(\omega_0\tau),$$

(20)

$$F_r \simeq eA_0 (1 - \beta_z\beta) \frac{\omega_0}{2} \tau \sin(\omega_0\tau),$$

(21)

where $A_0$ is a constant determined by the excitation mechanism and $c\beta_z$ is the axial velocity of a witness charged particle beam. Inside the channel, in the ultra-relativistic limit, the axial (accelerating) field is uniform with respect to transverse position as indicated by Eq. (20). Therefore electrons at different radii gain energy at the same rate, minimizing the energy spread due to the transverse extent of the beam. The transverse fields are linear with respect to the radial position as indicated by Eq. (21), which implies the root-mean-squared transverse normalized emittance will be conserved for any beam slice. Note that the focusing due to the excited fundamental mode fields is typically small in the ultra-relativistic limit (i.e., $|F_r/F_z| \sim 1 - \beta_z\beta \ll 1$). In addition, there is a $\pi/4$ phase region where the fundamental channel mode both focuses transversely and accelerates longitudinally. These properties make the hollow plasma channel well-suited for an accelerating structure independent of excitation mechanism.

LOSS FACTOR

The interaction of the beam with the accelerator environment can be quantified by a calculation of the loss factors. The loss factor per unit length $\kappa$ relates the accelerating gradient to the energy stored per unit length in the structure $U$ by $\kappa = E^2_z/4U$. The loss factor is purely geometrical and is independent of the excitation mechanism. Since the loss factor is independent of the means of energy deposition, it is a figure of merit for comparisons of accelerating structures.

The loss factor per unit length for the $m^{th}$ mode [13] is

$$\kappa_m = \frac{\omega_p^2}{c^2} \left[ \frac{K_m(R)}{RK_{m+1}(R)} \right] \left[ 1 + \frac{RK_m(R)}{2(m + 1)K_{m+1}(R)} \right]^{-1},$$

(22)
where the axial electric fields of the higher-order modes have been evaluated at the channel radius. The energy stored in the plasma structure $U_m$ is equal to the energy deposited by the driver. For an ultra-relativistic charge $q$ at a radius $r_b$, with $r_b < r_w$, the total energy deposited in the plasma structure by the charge can be written as

$$U = \sum_m U_m = \sum_m \kappa_m (r_b/r_w)^{2m} q^2.$$  

For comparison, the fundamental mode of a scaled disk-loaded copper SLC structure [14] has a loss factor of $\kappa_0^{(\text{SLC})} \approx 2.1 \times 10^3 \lambda_0^{-2} \text{(cm)} V/(\text{pC m})$, while the fundamental mode loss factor in a hollow plasma channel is $\kappa_0 = 3.6 \times 10^3 \lambda_0^{-2} \text{(cm)} K_0(R)/[R K_1(R)] V/(\text{pC m})$ where $\lambda_0 = \Omega_0^{1/2} c/\omega_p$ is the accelerating wavelength. For a normalized channel radius of $R = 1$, the fundamental mode loss factor is $\kappa_0 = 2.5 \times 10^3 \lambda_0^{-2} \text{(cm)} V/(\text{pC m})$, somewhat larger than the conventional resonantly-excited conducting structure, which implies stronger beam loading and smaller stored energy per unit length for a given accelerating gradient. Note that a larger loss factor results not only in greater energy deposition, but also in larger wake field excitation. The latter could result in instabilities and lead to greater energy spread.

**PARTICLE BEAM DYNAMICS IN A HOLLOW PLASMA CHANNEL**

The longitudinal and transverse forces on an ultra-relativistic beam due to its interaction with the plasma can be calculated from the convolution of the charge distribution of the beam with the wake fields produced by all proceeding charges. The wake fields excited inside the hollow plasma channel by a ultra-relativistic point charge $q$, passing through the channel at radius $r_b$, with $r_b < r_w$, and azimuthal angle $\theta = 0$, are

$$\vec{W}_\parallel = -q \sum_m \vec{W}_{||m}(\tau) r_m r_b^m \cos(m\theta) \hat{z},$$  

$$\vec{W}_\perp = q \sum_m \vec{W}_{\perp m}(\tau) r_m^{m-1} r_b^m [\hat{r} \cos(m\theta) - \hat{\theta} \sin(m\theta)],$$

with the wake functions,

$$\vec{W}_{||m}(\tau) = \frac{2\kappa_m}{r_{2m}} \cos[\Omega_m \omega_p \tau],$$  

$$\vec{W}_{\perp m}(\tau) = \frac{2m\kappa_m c}{r_{2m} \Omega_m \omega_p} \sin[\Omega_m \omega_p \tau].$$

The mode frequencies, $\Omega_m$, and loss factors, $\kappa_m$, are given by Eqs. (19) and (22), respectively. The Laplace transform of the wake functions Eqs. (26) and (27)
yields the impedance of the plasma structure. These point charge wake fields (Eqs. (24) and (25)) can be used as Green functions to compute the longitudinal and transverse forces produced by an arbitrary beam charge distribution, $\rho_b$. The longitudinal wake fields tend to cause energy variation within a bunch, and the transverse wake fields cause beam breakup instabilities. In the case that the charge is near the axis of the channel, $r_b \ll r_w$, the longitudinal wake field is dominated by the fundamental ($m = 0$) mode and the transverse wake field is dominated by the dipole ($m = 1$) mode.

### Longitudinal Effects

The longitudinal wake fields will cause the energy spread $\sigma_\gamma$ within the beam to grow. Energy spread constraints will therefore limit the beam current. The energy change of an ultra-relativistic electron bunch propagating along the axis of the hollow plasma channel is described by the equation

$$\frac{\partial \gamma}{\partial z} = c I^{-1}_0 E(\tau) - \int_0^\tau c I^{-1}_0 I(\zeta) \hat{W}_{||0}(\tau - \zeta) d\zeta,$$

where $\hat{W}_{||0}$ is the longitudinal fundamental mode wake function given by Eq. (26). Here, $E(\tau) = A_0 \cos(\omega_0 \tau - \varphi_{\text{inj}})$ is the accelerating gradient, with $A_0$ the peak axial electric field of an excited plasma wave (created by a drive pulse) and $\varphi_{\text{inj}}$ the injection phase of the head of the bunch with respect to the plasma wave. For a delta function bunch $I(\tau) = q \delta(\tau)$ (i.e., a bunch much shorter than the period of the accelerating field $\omega_p \tau_b \ll 1$), one finds $\sigma_\gamma/\gamma \approx q \hat{W}_{||0}(0)/[2E(0)] = q \kappa_0/E(0)$. For illustration, if an energy spread of order 0.1% is required in a plasma structure with $r_w = 20 \mu$m, $R = 1$, and an accelerating gradient of 10 GV/m, then the beam-induced gradient should be held to $2\kappa_0 q \approx 20$ MV/m. The single-bunch charge $q$ is then limited to 0.9 pC or $5 \times 10^6$ particles. In principle, the energy spread within a single bunch can be minimized and the charge limits increased by shaping the charge distribution of the bunch [15], although this may be difficult to achieve in practice.

### Transverse Instabilities

It is well-known in accelerator physics that interaction of the beam with the structure geometry can result in transverse instabilities coupled to the off-axis displacement of the beam centroid, or beam breakup instabilities [16]. This section discusses beam breakup instabilities in a hollow plasma channel.

Consider the effect of a small displacement of the beam centroid $X(z, \tau)$ in the transverse direction. The transverse displacement is expressed as a function of two variables: the propagation distance, $z$, and the distance from the head of the beam, $\tau$. The variable $\tau = t - z/v_b$ indexes beam slices where $v_b \simeq c$ is the axial beam
velocity. The beam extends from $\tau = 0$ (the beam head) to $\tau = \tau_b$ (the beam tail). Beam electrons remain approximately at a fixed $\tau$, as they advance in $z$ along the length of the accelerator.

From the Lorentz force equation, assuming the beam is monoenergetic, the evolution of the transverse displacement of the beam due to the dipole transverse wake field is

$$\left[ \frac{\partial}{\partial z} \gamma(z) \frac{\partial}{\partial z} + \gamma(z) k_{p}^{2}(z) \right] X(z, \tau) = \int_{0}^{\tau} cI(\zeta) I_{o}^{-1} \hat{W}_{11}(\tau - \zeta) X(z, \zeta) d\zeta, \quad (29)$$

where $I(\tau)$ is the beam current and $I_o = m_e c^3/e \approx 17 \text{ kA}$ is the Alfvén constant. The transverse dipole wake function $\hat{W}_{11}$ is given by Eq. (27) with $m = 1$ and determines the Lorentz force on an electron at $\tau$ as it arrives at $z$ due to the fields generated by the beam segment at $\zeta < \tau$. The right-hand side of Eq. (29) is the cumulative force due to the transverse dipole wake fields of the proceeding charges in the beam. The transverse focusing force in the channel from a plasma wave (created by a drive pulse) and from any external magnets can be described, in the linear approximation, by the betatron wavenumber $k_p(z)$. This model [i.e., Eq. (29)] is valid in the ultra-relativistic limit of the beam velocity, where phase slippage between particles in the bunch is small. Equation (29) can be solved in a variety of limits to study the single-bunch beam breakup instability.

### Single-Bunch Beam Breakup

Assuming the beam density, which is proportional to the beam current, remains constant, Eq. (29) can be rewritten as

$$\left[ \frac{\partial}{\partial z} \gamma(z) \frac{\partial}{\partial z} + \gamma(z) k_{p}^{2}(z) \right] X(z, \tau) = \int_{0}^{\tau} d\zeta G(\tau - \zeta) X(z, \zeta), \quad (30)$$

where the Green function $G$ is given by the excited wake field,

$$G(\tau) = c I I_{o}^{-1} \hat{W}_{11}(\tau) = \left( \frac{I}{I_{o}} \frac{2\kappa_{1}\omega_{p}}{R^{2}\Omega_{1}} \right) \sin(\omega_{1}\tau) = G_{1}\sin(\omega_{1}\tau). \quad (31)$$

In the limit of a short bunch (i.e., $\omega_{p}\tau_{b} \ll 1$), the wake field response is approximately linear $G(\tau) \simeq (G_{1}\omega_{1})\tau$.

If the growth length of the instability is much less than $k_{p}^{-1}$ (i.e., the weak-focusing regime), then the term due to transverse focusing on the left-hand side of Eq. (30) can be neglected. This will typically be valid for ultra-relativistic beams propagating in hollow plasma channels without external focusing since the transverse focusing forces in the channel due to the excited fundamental mode fields will be small in the ultra-relativistic limit, as indicated by Eq. (21) with $(1 - \beta z \beta) \simeq 0$. 

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Equation (30) can be solved following the method in Ref. [17] (i.e., by assuming that the growth of the transverse beam displacement and the change in energy to be slow on the scale of the accelerating field). In this approximation Eq. (30) has the solution

\[ X(z, \tau) = X_0 \left( \frac{\gamma_0}{\gamma} \right)^{1/4} \frac{1}{2\pi i} \int_{-\infty}^{\infty} ds \exp \left[ s \tau \pm \frac{\sqrt{G_1 \omega_1}}{s} \int_0^z dz_1 \gamma^{-1/2}(z_1) \right], \quad (32) \]

where \( X_0 \) is the initial transverse displacement of the beam. Applying the method of steepest descent to the integral in Eq. (32), one finds

\[ X(z, \tau) \approx X_0 \left( \frac{\gamma_0}{\gamma} \right)^{1/4} \frac{\exp(\Lambda_w)}{\sqrt{8\pi \Lambda_w}}, \quad (33) \]

with the exponent

\[ \Lambda_w = 2 \left[ \left( G_1 \omega_1^2 \right)^{1/2} \int_0^z dz_1 \gamma^{-1/2}(z_1) \right]^{1/2}. \quad (34) \]

For an ultra-relativistic beam, longitudinal slippage is negligible the beam slices remain at fixed \( \tau \). With fixed \( \tau \) and \( \omega_p \tau_b \ll 1 \), the axial force Eq. (20) is constant along the beam and the energy growth is linear. Thus, \( \gamma = \gamma_0 + gz \), where \( \gamma_0 \) is the initial energy and \( g \) is the constant acceleration gradient. The exponent \( \Lambda_w \) becomes

\[ \Lambda_w = 2^{3/2} \left( G_1 \omega_1^2 \frac{z}{g^2} \right)^{1/4} \left( \gamma^{1/2} - \gamma_0^{1/2} \right)^{1/2}. \quad (35) \]

Asymptotically (for large \( z \)), the exponent Eq. (35) is

\[ \Lambda_w \to 2^{3/2} \left( G_1 \omega_1^2 \frac{z}{g^2} \right)^{1/4} = \left( \frac{z}{L_w} \right)^{1/4}, \quad (36) \]

with the characteristic growth length

\[ L_w = g \left( 2^6 G_1 \omega_1^2 \right)^{-1} = 2^{-7/2} \frac{L_0}{\kappa_1 (\omega_p \tau)^2} \frac{g R^2}{I \kappa_1 (\omega_p \tau)^2}. \quad (37) \]

This growth length will impose an upper bound on the accelerator length for a given \( I \tau_b^2 \) product. For example, in a plasma channel with plasma wavelength of 125 \( \mu \)m, channel radius of 20 \( \mu \)m, and accelerating gradient of 10 GV/m, a 3 fs beam with a charge of 1 pC will have an instability growth length of \( L_w \approx 5 \) mm. As Eq. (37) indicates, the instability growth length can be increased by increasing \( R \), which, in turn, will lower the loss factor of the structure (assuming a fixed plasma density).

The asymptotic growth of the transverse displacement of the beam centroid can also be determined for a particle beam which is traveling through an unexcited
(\(g = 0\)) hollow plasma channel (i.e., a coasting beam or drive beam). In this case, the exponent from Eq. (35) becomes

\[
\Lambda_w = 2 \left( \frac{G_1 \omega_1 \gamma_0^2}{\gamma_0} \right)^{1/4} z^{1/2} = \left( \frac{z}{L_{wu}} \right)^{1/2},
\]

with the characteristic growth length

\[
L_{wu} = \frac{1}{4 \pi} \left( \frac{\gamma_0}{G_1 \omega_1} \right)^{1/2} = 2^{-5/2} \left[ \frac{I_e}{\kappa_1} \frac{\gamma_0 R^2}{(\omega_p \gamma)^2} \right]^{1/2}.
\]

As one can see from comparison of the growth lengths Eqs. (37) and (39), acceleration tends to reduce the influence of transverse beam displacements.

**Single-Bunch Beam Breakup with External Focusing**

The transverse focusing in the hollow plasma channel provided by the accelerating (fundamental) wake field is weak for relativistic beams. For high-energy applications one may prefer to provide external focusing to operate in the strong-focusing regime (i.e., \(k_p^2 \ll 1\) is much less than the instability growth length). As shown below, external focusing (e.g., magnetic quadrupole lens) can substantially reduce the asymptotic growth of transverse beam displacements.

The asymptotic growth of the transverse centroid displacement of an accelerated and strongly focused beam can be determined by applying an eikonal approximation to Eq. (30), assuming the growth will be slow on the scale of a betatron period. Consider the slowly varying amplitude of the transverse centroid displacement \(\chi(z, \tau)\) such that

\[
\chi(z, \tau) = \left( \frac{\gamma \dot{\theta}_0}{\gamma \dot{\theta}_\beta} \right)^{1/2} \chi(z, \tau) \exp[i \theta_\beta(z)],
\]

with the betatron phase

\[
\theta_\beta(z) = \int_0^z dz_1 k_\beta(z_1)
\]

and \(\dot{\theta}_0 = \dot{\theta}_\beta(z = 0)\). Substituting Eq. (40) into Eq. (30), assuming the eikonal approximation such that \(|\dot{\gamma}| \ll |\gamma \dot{\theta}_\beta|\) and \(|\dot{\chi}| \ll |\chi \dot{\theta}_\beta|\), and taking a Laplace transform in \(\tau\) yields

\[
\frac{\partial}{\partial z} \tilde{\chi}(z, s) = \frac{\tilde{G}}{2i \gamma \dot{\theta}_\beta} \tilde{\chi}(z, s),
\]

which has the solution
where \( \bar{X}_0 = \bar{X}(z = 0, s) \). Inverting the Laplace transform, the solution for the amplitude of the beam centroid is

$$
\bar{X}(z, s) = \bar{X}_0 \exp \left[ \frac{\tilde{G}(s)}{2i} \int_0^z \frac{dz_1}{\gamma(z_1) \theta_\beta(z_1)} \right],
$$

(43)

where the initial condition \( \bar{X}(z = 0, \tau) = X_0 \Theta(\tau) \) is assumed. The integral in Eq. (44) may be computed approximately by the method of steepest descent. Using this method, one finds the transverse beam displacement is

$$
X(z, \tau) \approx X_0 \frac{3^{1/4}}{2^{3/2}\pi^{1/2}} \left( \frac{\gamma_0 \dot{\theta}_0}{\gamma \dot{\theta}_\beta} \right)^{1/2} \text{exp} \left( \frac{\Lambda_s}{\Lambda_0^{1/2}} \right) \cos \left( \theta_\beta - \frac{\Lambda_s}{\sqrt{3}} + \frac{\pi}{12} \right),
$$

(45)

with the exponent

$$
\Lambda_s = \frac{3^{3/2}}{4} \left[ \frac{G_1 \omega_1 \tau^2}{\rho} \right] \left[ \int_0^z \frac{dz_1}{\gamma(z_1) \theta_\beta(z_1)} \right]^{1/3}.
$$

(46)

In deriving Eq. (45), a short bunch, \( \omega_p \tau_b \ll 1 \), was assumed such that \( G(\tau) \simeq G_1 \omega_1 \tau \).

Considering linear energy growth \( \gamma(z) = \gamma_0 + g z \) and assuming the betatron wavenumber has an energy dependence such that \( k_\beta(z) = \dot{\theta}_\beta = \theta_0 (\gamma_0 / \gamma)^{\alpha} \), the transverse beam displacement of a short bunch becomes

$$
\frac{X(z, \tau)}{X_0} \approx \frac{3^{1/4}}{2^{3/2}\pi^{1/2}} \left( \frac{\gamma_0}{\gamma} \right)^{(1-\alpha)/2} \text{exp} \left( \frac{\Lambda_s}{\Lambda_0^{1/2}} \right) \cos \left( \theta_\beta - \frac{\Lambda_s}{3^{1/2}} + \frac{\pi}{12} \right),
$$

(47)

with the betatron phase

$$
\theta_\beta = \frac{\gamma_0^{\alpha} k_0}{g(1-\alpha)} \left( \gamma^{1-\alpha} - \gamma_0^{1-\alpha} \right)
$$

(48)

and exponent

$$
\Lambda_s = \frac{3^{3/2}}{2^{3/3}} \left[ \frac{I_0}{\kappa_1 (\omega_p \tau)^2} \right]^{1/3} \left( \gamma^\alpha - \gamma_0^\alpha \right),
$$

(49)

where \( k_0 = k_\beta(z = 0) \) is the initial betatron wavenumber at injection. Asymptotically, \( \Lambda_s \rightarrow (z/L_\alpha)^{\alpha/3} \), with the instability growth length

$$
L_\alpha = \frac{2^{5/2}}{3^{9/2\alpha}} \left( \frac{I_0}{I} \right)^{1/\alpha} \left[ \frac{\alpha g^{1-\alpha} \gamma_0^{\alpha} k_0 R^2}{\kappa_1 (\omega_p \tau)^2} \right]^{1/\alpha}.
$$

(50)
For example, if $\alpha = 1/2$ (e.g., a magnetic quadrupole lens), then the growth rate scales as $L_{s} \propto (I/I_o)^{-2}(\omega_p \tau)^{-4}$, which is a more favorable scaling than Eq. (37).

For a coasting or drive beam [i.e., no acceleration ($g = 0$)] traveling through an unexcited hollow plasma channel structure with strong external focusing, $\gamma(z) = \gamma_0$, $\theta(z) = k_{\beta}z$, and

$$\Lambda_{s} = \frac{3^{3/2}}{2^{5/3}} \left( \frac{G_{1}\omega_{1} \tau^2 z}{2k_{\beta} \gamma_{0}} \right)^{1/3} = \left( \frac{z}{L_{su}} \right)^{1/3},$$

where the characteristic growth length is

$$L_{su} = \frac{2^{6}k_{\beta} \gamma_{0}}{3^{9/2}G_{1}\omega_{1} \tau^2} = \frac{2^{5}}{3^{9/2}} \frac{I_{o}}{I} \frac{k_{\beta} \gamma_{0} R^{2}}{\kappa_{1}(\omega_{p} \tau)^{2}}.$$

### CONCLUSIONS

This paper characterizes the hollow plasma channel in terms of fundamental accelerator parameters: the mode frequencies (eigenvalues) and the loss factors (eigenfunctions) of the electromagnetic channel modes excited by a driver (laser or particle beam) of arbitrary velocity. With these results, one can quantify the performance of a high-energy machine based on this plasma structure. In order to reach TeV-energies, such a plasma-based accelerator would consist of many stages. With optical guiding provided by the plasma channel, the length of a single stage based on a hollow plasma channel structure would be fundamentally limited by the shorter of the dephasing length and the driver depletion length. In practice, the length of a plasma-based accelerator may be limited by beam-plasma instabilities.

Plasmas provide strong coupling for acceleration of particle beams, quantified in the loss factors. At the same time this strong coupling extends to strong deflection and breakup of the beam. In this paper, the stability of a particle beam propagating in a hollow plasma channel was examined. We addressed the coupling of the dipole wake field excited in the plasma channel to the transverse displacement of the beam. Single-bunch beam breakup was analyzed for accelerating and coasting beam propagation in a hollow plasma channel in the weak-focusing and the strong-focusing regimes. These results show that the most favorable scalings are achieved for beams propagating in an excited hollow plasma channel in the strong-focusing regime.

With diffraction overcome by a plasma channel, the most severe limitation to the length of a single accelerator stage based on the hollow plasma channel structure is the transverse stability of the particle beam. That is, for typical parameters of plasma-based accelerator experiments the transverse beam breakup growth length will be shorter than both the dephasing lengths or the driver depletion lengths. As it does in the study of high-energy accelerators based on conventional rf technology, the transverse instabilities will set limits on allowable jitter and alignment tolerances. Using the analysis presented in this paper, such limits can be meaningfully estimated for the hollow plasma structure.
ACKNOWLEDGEMENTS

The authors thank Prof. David H. Whittum for useful discussions. This work was supported by the US Department of Energy, Division of High-Energy Physics.

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