Green function analysis of a Raman FEL

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Abstract

This paper derives, in closed form, the Green function of an FEL operating in the strongly Raman regime. This Green function allows for the calculation of the temporal and spacial evolution of an arbitrary input radiation pulse. For the first time superradiance, originally studied in Compton regime by Bonifacio and co-workers [Phys. Rev. Lett. 73 (1994) 70; Nucl. Instr. and Meth. A 239 (1985) 36], has been seen numerically in a strongly Raman FEL.

1. Introduction

The desire to build compact free-electron lasers and to develop high-power microwave sources often requires operation in the Raman regime. To date many of the experiments in the Raman regime have not had significant effects from slippage. There are applications, such as RF accelerator power sources, where detailed knowledge and control of the phase and amplitude of the free-electron laser (FEL) output is of critical importance. With the improving performance of photocathode guns, FEL experiments, using RF linacs as drivers, with modest beam energies and high currents, may operate in the high gain Raman regime with strong slippage.

Most theoretical research for the free-electron laser (FEL) operating in the Raman regime has concentrated on either the linear growth in an amplifier, the derivation and solution of nonlinear equations for the self-consistent coupling between particles and fields, and comparison of simulations with experiments. The desire to build compact FELs and to develop high-power microwave sources often requires operation in the Raman regime.

The approach used here is to first formulate the linear FEL equations, including space-charge, in integral form, solve the equations by Fourier–Laplace transform in the limit of a strongly Raman FEL and perform an inverse transform to find the Green function. An exact expression for the Raman regime Green function is obtained in the limit of a long electron pulse. The expression is fully causal, is valid for high and low gain and includes all launching losses. Effects similar to Compton regime superradiance are found.

2. Integral equation

This paper follows the formalism of Ref. [1], where the normalized independent variables are

$$\tilde{z} = \frac{2\gamma_0^2 c}{N_w \lambda_w} \left( t - \frac{z}{v_0} \right),$$

(1)

$$\tilde{z} = \frac{z}{(N_w \lambda_w)}.$$  

These variables are chosen so that $$\tilde{z}$$ propagates with the optical pulse and is measured in units of the slippage length, and interaction length $$z$$ is normalized to the wiggler length; thus $$d\tilde{z}/dz = 1$$. The free-streaming phase of a particle in the ponderomotive wave is given by

$$\theta_0 = 2\pi N_w (\tilde{z} - \tilde{s}) + \tilde{s} (k_w - k_\gamma^2/2\gamma_0^2) N_w \lambda_w,$$

(2)

where $$N_w$$ is the number of wiggler periods, $$\lambda_w$$ is the wiggler period, $$\gamma_0$$ is the electron relativistic factor calculated from the parallel velocity, $$c$$ is the speed of light, and $$v_0$$ is the group velocity of light and the normalized detuning over the wiggler is

$$\gamma_0 (k_w - k_\gamma^2/2\gamma_0^2) N_w \lambda_w.$$

Within the context of a cold beam fluid model valid prior to saturation, the FEL evolution equations for an arbitrary gain during a single pass through the wiggler can be written as:

$$\frac{d\tilde{a}}{d\tilde{z}} = \frac{2\pi i a_w Ler \delta n \exp(-i\theta)}{\gamma k},$$

(4)

$$\frac{d^2\theta}{d\tilde{z}^2} = i \frac{8\pi^2 N_w^2 a_w}{1 + a_w^2} \tilde{s} \tilde{n} \exp(i\theta)$$

$$- i \frac{\omega_0^2 N_w^2 \gamma_0^2}{c^2 \gamma_0^2} \delta n / n_0,$$

(5)

$$y_0 = \frac{d\theta}{d\tilde{z}} (\tilde{z} = 0), \quad \theta_0 = \theta (\tilde{z} = 0).$$

(6)

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In this notation the normalized slowly varying complex amplitude of the field is $\hat{a}(\tilde{s}, \tilde{z})$, the detuning is $\gamma_0$, the Lagrangian phase variable is $\theta$, the wigglar amplitude is $a_w$, $L_w = N_w \lambda_w$, $k = 2\pi/\lambda$ is the radiation wavenumber, $\omega_w = 4\pi c^2 n_0 / m$ is the non-relativistic beam plasma frequency, and the density perturbation $\delta n$ can be expressed in terms of the phase as:

$$\delta n = -n_0 (\tilde{s} - \tilde{z}) \left( \frac{\partial \theta(\theta_0, \tilde{z})}{\partial \theta_0} - 1 \right),$$

where $n_0$ is the beam density. Note that $1 - \tilde{z}/v_z = (N_w \lambda_w / c)(\tilde{s} - \tilde{z})$. We will assume here that the unperturbed beam density $n_0$ is constant. A related analysis with a nonuniform beam density is presented in another paper [2]. The second term in Eq. (5) is due to the space-charge force.

In linear theory, the phase can be expanded as:

$$\theta = \theta_0 + \gamma_0 \tilde{z} + \int_0^\tilde{z} d\tilde{z}' \int_0^{\tilde{z}'} d\tilde{z}'' \left( \frac{8\pi^2 a_w N_w^2}{1 + a_w^2} \exp\left[i(\gamma_0 \tilde{z}' + \theta_0)\right] \hat{a}(\tilde{s} - \tilde{z} + \tilde{z}' + \tilde{z}'') \right) \times \exp\left[i(\gamma_0 \tilde{z}' - \omega)(\tilde{s}' - \tilde{z})\right] \left( \frac{\omega_0^2 N_w^2 k^2}{c^2 \gamma_0 \gamma_c^2} \frac{\delta n}{n_0} (\tilde{s} - \tilde{z} + \tilde{z}' + \tilde{z}'') \right).$$

Note that $\tilde{s} - \tilde{z} = \tilde{s}_0$ is the Lagrangian entrance time of a particle into the wiggler.

An integro-differential equation for the field can be obtained:

$$\frac{\partial \hat{a}(\tilde{s}, \tilde{z})}{\partial \tilde{z}} + \frac{\omega_0^2 N_w^2 k}{c^2 \gamma_0 ^2} \int_0^{\tilde{s}} d\tilde{s}' \int_0^{\tilde{s}'} d\tilde{s}'' \frac{\partial \hat{a}(\tilde{s} - \tilde{z} + \tilde{s}' + \tilde{s}'')}{\partial \tilde{s}''}$$

$$\times \exp\left[i(\gamma_0 \tilde{z}' - \omega)(\tilde{s}' - \tilde{z})\right] \times \exp\left[i(\gamma_0 \tilde{z}' - \omega)(\tilde{s}' - \tilde{z} + \tilde{s}'' + \tilde{s}''')\right]$$

$$= i j_e \int_0^{\tilde{s}} d\tilde{s}' \int_0^{\tilde{z}'} d\tilde{z}'' \hat{a}(\omega, \tilde{z}'') \times \exp\left[i(\gamma_0 - \omega)(\tilde{s}' - \tilde{z})\right],$$

and then a Laplace transform in $\tilde{z}$ using

$$\hat{a}(\omega, \Gamma) = \int_0^{\infty} d\tilde{z} \exp\left[i \Gamma \tilde{z}\right] \hat{a}(\omega, \tilde{z})$$

with the requirement that the initial field derivatives vanish:

$$\frac{\partial \hat{a}(\tilde{z} = 0, \tilde{s})}{\partial \tilde{z}} - \frac{\partial^2 \hat{a}(\tilde{z} = 0, \tilde{s})}{\partial \tilde{s}^2} = 0.$$

The result can be found in the literature (for example, a Green function approach was used to study start-up in a Raman regime oscillator [4] and superradiance and growth from noise in a high gain amplifier [5]) so we omit the details here. The main point of this paper is to show that for a strongly Raman FEL a closed form for the Green function can be derived and to study Raman superradiance.

After an inverse Laplace transform, one obtains:

$$\hat{a}(\omega, \tilde{z}) = j_e \hat{a}(\omega, \tilde{z} = 0) \left( \Gamma_1 \left( \Gamma_1 - \Gamma_2 \right) \left( \Gamma_1 - \Gamma_3 \right) \right)$$

$$+ \Gamma_2 \left( \Gamma_2 - \Gamma_3 \right) \left( \Gamma_2 - \Gamma_1 \right)$$

$$+ \Gamma_3 \left( \Gamma_3 - \Gamma_2 \right) \left( \Gamma_3 - \Gamma_1 \right),$$

where the roots $\Gamma_i$ satisfy the cubic dispersion relation:

$$\Gamma \left( \Gamma + \gamma_0 - \omega \right)^2 - k_0^2 = -j_e.$$   

(14)

Here the relativistic beam plasma frequency is $k_0^2 = N_w^2 \lambda_w^2 \omega_0^2 c^2 \gamma_c^2$. The usual steady-state complex gain is recovered by taking $\omega = 0$.

For an FEL in the strong Raman regime the above cubic reduces to a quadratic, in the usual way, by assuming the detuning is near to the slow space-charge wave and the growth rate is much smaller than the beam plasma frequency. In this approximation,

$$\Gamma \left( \Gamma + \gamma_0 - \omega \right)^2 - k_0^2 = -\mu^2/4,$$   

(15)

where $\mu^2 = 2j_e/k_b$ and $Y = \gamma_0 - \omega$. The Green function is then:

$$G_s(\tilde{s}) = \frac{\mu^2}{4} \int_{-\infty + i\delta}^{\infty + i\delta} d\omega \exp(i \omega \tilde{s})$$

$$\times \left\{ \Gamma_1(\gamma_0 - \omega) \left( \Gamma_1(\gamma_0 - \omega) - \Gamma_2(\gamma_0 - \omega) \right) + \Gamma_2(\gamma_0 - \omega) \left( \Gamma_2(\gamma_0 - \omega) - \Gamma_1(\gamma_0 - \omega) \right) \right\},$$

(16)
and the roots
\[ I_1 = \frac{y - k_b}{2} + \frac{1}{2} \sqrt{(y - k_b)^2 - \mu^2}, \]
\[ I_2 = \frac{y - k_b}{2} - \frac{1}{2} \sqrt{(y - k_b)^2 - \mu^2}. \]

4. Explicit Green function

Carefully performing the inverse Laplace transform (Eq. (16)) yields an explicit expression for the Green function for the field evolution from a given initial field, and a parallel analysis yields the evolution of the field from the input noise in \( \delta n \). The result is

\[ a'(\bar{z}, \bar{s}) = a'(\bar{s}) \left| \bar{s} = 0 \right. + \int_{\bar{s} = \bar{z}}^{\infty} d\bar{s}' G_a(\bar{z}, \bar{s} - \bar{s}') a'(\bar{s}') \right| \bar{s} = 0 \]
\[ + \frac{2 \pi i \omega \epsilon r_e}{\gamma k} \int_{\bar{s} = \bar{z}}^{\infty} d\bar{s}' G_a(\bar{z}, \bar{s} - \bar{s}') \delta n(\bar{s}') \]
\[ (19) \]

with the Green functions \( G_a \) and \( G_n \) given by

\[ G_a(\bar{z}, \bar{s}) = \frac{\mu}{2} (\bar{s} - \bar{z})^{1/2} \bar{s}^{1/2} \]
\[ \times \exp(i\bar{s}(k_b - y_0)) I_1(\mu \bar{s}^{1/2}(\bar{z} - \bar{s})^{1/2}), \]
\[ (20) \]

\[ G_n(\bar{z}, \bar{s}) = \exp(i\bar{s}(k_b - y_0)) I_0(\mu \bar{s}^{1/2}(\bar{z} - \bar{s})^{1/2}). \]
\[ (21) \]

Here \( I_1 \) and \( I_0 \) are the usual modified Bessel functions. As expected from causality, both functions vanish for \( \bar{s} < 0 \) and \( \bar{s} > \bar{z} \). In the absence of initial density perturbation, and in the limit that the coupling goes to zero, i.e., \( \mu = 0 \), the field amplitude propagates unaltered for any pulse shape (since \( G_n = 0 \)). The expression is fully causal, is valid for high and low gain and includes slippage, all launching losses, and start up from noise (SASE in the Raman regime).

We can recover the steady-state case with an initially time independent radiation field of the form

\[ a(\bar{s}, \bar{z} = 0) = a_0, \]

and no density perturbation. The steady-state amplified field can be found, using some well known Bessel function relations, to be:

\[ a(\bar{s}, \bar{z}) = a_0 \cosh(\mu \bar{z}/2). \]

5. Superradiance

Superradiance has been studied in the Compton regime [5]. It is also present in the Raman regime. Superradiant spikes are seen when an initially flat-top pulse enters the wiggler detuned out of the growth bandwidth. The pulse evolution is shown in Figs. 1 and 2. In Fig. 1 the initial field amplitude is shown at the entrance to the wiggler. The gain corresponds to \( \mu = 10 \) and \( y_0 - k_b = 10.5 \), or a 5% detuning from the gain bandwidth around the slow space-charge wave. Thus in steady state there would be no exponential growth. In Fig. 2 the amplitude and frequency shift (with respect to the initial detuning) at the end of the wiggler are plotted. As can be seen, superradiant spikes have formed at the pulse edges and have grown by frequency shifting the initial pulse into the gain bandwidth (peak gain corresponds to a frequency shift of 10.5 in Fig. 2). This shifting is gradual, rather than instantaneous – the
superradiant process involves both growth and frequency shifts. The peak gain frequency is reached only asymptotically, after many $-\varepsilon$-foldings.

The calculated Green functions are responses to delta-function perturbations. One can show that the maximum of the Green function occurs at $\tilde{z} = \tilde{z}/2$, corresponding to a group velocity of $v_g = (c + v_p)/2$. This is a manifestation of laser lethargy in the strongly Raman regime.

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References