Beam Envelope Equations for Cooling of Muons in Solenoid Fields

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(Received 22 March 2000)

Muon cooling is a critical component of the proposed muon collider and neutrino factory. Previous studies of cooling channels have tracked single muons through the channel, which requires many particles for good statistics and does not lend itself to an understanding of channel dynamics. In this paper, a system of moment equations are derived which captures the major aspects of cooling: interactions with material and acceleration by radio frequency (rf) cavities. A general analysis of solenoid lattice types compares well with prior simulations and indicates new directions for study.

PACS numbers: 41.85.Lc

There has been significant research over the past five years on the feasibility of building a muon collider [1] and, more recently, growing excitement over the possibility of building a muon storage ring neutrino source [2,3]. Both of these concepts envision an intense proton beam incident on a target and producing pions, which then decay into muons. The production of muons is expensive and a substantial target and producing pions, which then decay into muons.

In beam phase space due to scattering is minimized by ensuring that the angular spread of velocities in the beam is as large as possible; this corresponds to focusing the beam to a small spot size. As the beam cools transversely, an unfavorable variation of energy loss with energy $d(dP/ds)/dP$ creates longitudinal heating. Eventually, particles can no longer be accelerated by the rf, and begin to be lost.

The method proposed to overcome this problem is emittance exchange, in which the six-dimensional phase space is manipulated so that some longitudinal phase space is added to the transverse space. The now larger transverse phase space can be reduced through ionization cooling. While there are various suggestions for implementing emittance exchange, none have been proven workable in any realistic situation. Present designs for the neutrino factory do not require emittance exchange—the phase space reduction of around 100 can be fully transverse. This paper studies the dynamics of transverse cooling and is restricted to azimuthally symmetric beams. Extensions to include nonaxisymmetric channels are straightforward and are likely to be needed for the study of emittance exchange as well as of beam lines with bends and magnet errors.

We first consider single particle motion in vacuum magnetic fields. The magnetic field inside of a cylindrically symmetric solenoid is $B = \nabla \times [\mathcal{A}_\phi (r, z) \dot{\phi}]$. For a paraxial theory, it is sufficient to approximate $\mathcal{A}_\phi \approx r B(z)/2$, where $B(z) \equiv B_c (r = 0, z)$. The constants of motion are $P^2$ (or energy) and the canonical angular momentum, $L_{\text{canon}} = x P_y - y P_x + q r \mathcal{A}_\phi$, where $q$ is the charge. In a rotating coordinate frame (the Larmor frame), with $X_R = x \cos \varphi - y \sin \varphi$, $Y_R = x \sin \varphi + y \cos \varphi$, and

$$\varphi' = \frac{q A_{\phi}}{P_z r} = \frac{q B(z)}{2 P_z} \equiv \kappa,$$

the linearized equations of motion reduce to $X_R' = -\kappa^2 X_R$ and $Y_R' = -\kappa^2 Y_R$; here, $\prime$ indicates the derivative along the $z$ axis.

We can parametrize the solutions to the linearized equations in terms of a “betatron function,” $\beta_p$, and phase, $\Phi$, by $X_R = A_1 \sqrt{\beta_p} \cos (\Phi - \Phi_1)$ and $Y_R = A_2 \sqrt{\beta_p} \cos (\Phi - \Phi_2)$. The transverse amplitudes $A_1$ and $A_2$ correspond to the Courant-Snyder invariants [5]. The betatron function satisfies $\Phi' = 1/\beta_p$, and $\beta_p$ evolves as

$$2 \beta_p \beta_p' - (\beta_p')^2 + 4 \beta_p^2 \kappa^2 - 4 = 0.$$ 

The angular momentum is, to lowest order, $L_{\text{canon}} = P_z A_1 A_2 \sin (\Phi_2 - \Phi_1)$. The forward momentum is then determined by $P^2 = P_z^2 [1 + (x')^2 + (y')^2].$
Now consider a cylindrically symmetric distribution of particles in transverse phase space. Our goal is to find a simple description of the rms properties of the particle distribution. These are described by the matrix $M$ of the second-order transverse beam moments. For simplicity, we neglect coupling between longitudinal and transverse moments. The matrix $M$, which is also the covariance matrix for the axis-centered beam, contains, by symmetry, only four independent moments in $x$, $P_x$, $y$, and $P_y$, and

$$\frac{M}{mc} = \begin{pmatrix} \frac{P_z}{\beta_\perp} & 0 & \frac{P_z\gamma_\perp}{\beta_\perp - \beta_\perp \kappa} & 0 \\ 0 & \frac{P_z\gamma_\perp}{\beta_\perp - \beta_\perp \kappa} & \frac{P_z\gamma_\perp}{\beta_\perp - \beta_\perp \kappa} & 0 \\ \frac{\alpha_\perp}{\beta_\perp} & 0 & -\alpha_\perp & 0 \\ 0 & -\alpha_\perp & 0 & -\alpha_\perp \end{pmatrix}.$$  

Here, the average value of $P_z$ is used, and we set $\kappa = qB(z)/2(P_z)$. The definition of emittance combined with Eq. (3) imposes the condition

$$\gamma_\perp = \frac{1}{\beta_\perp} [1 + \frac{\alpha_\perp}{\beta_\perp} + (\beta_\perp \kappa - \mathcal{L})^2].$$  

Starting from the deterministic single particle equations, $x' = P_x/P_z$, $y' = P_y/P_z$, and

$$v_z \frac{dP_z}{dz} = \frac{dP}{dt} = q(\ddot{E} + \ddot{v} \times \dot{B}) + \ddot{v} \frac{dP}{ds},$$  

where $dP/ds < 0$ is the average change in momentum per path length in material, the equations of evolution can be found by interchanging the operations of averaging and differentiation so that, for example, $d(x^2)/dz = 2xP_x/P_z$. Then, to incorporate the effect of multiple scatter, an additional term $P_x \partial_\perp$ is added to the derivatives of $\langle P^2_x \rangle$ and $\langle P^2_z \rangle$; here, $\mathcal{S} = d(x^2)/dz = (13.6 \text{ MeV}/\nu P)^2/LR$, and $L_R$ is the radiation length of the material, using a Gaussian fit [6] to the Molière theory for multiple scatter.

Assuming that correlations between longitudinal and transverse quantities are weak (neglecting energy dispersion, for example), and considering in the transverse magnetic field only $B_z \approx r$, the evolution of the beam is determined by

$$\epsilon_N' = \beta_\perp \frac{PS}{2mc} + \epsilon_N \frac{1}{P_z} \frac{dP_z}{dz},$$  

$$\beta_\perp' = -2\alpha_\perp + \beta_\perp \frac{q(E_z)}{v_z P_z} - \frac{P_z^2}{\epsilon_N} \frac{PS}{2mc},$$  

$$\alpha_\perp' = -\gamma_\perp + 2\kappa(\beta_\perp \kappa - \mathcal{L}) - \frac{\alpha_\perp \beta_\perp}{\epsilon_N} \frac{PS}{2mc},$$  

$$\mathcal{L}' = -\beta_\perp \kappa \frac{1}{P_z} \frac{dP_z}{dz} - \frac{\mathcal{L} \beta_\perp \frac{PS}{2mc},}{\epsilon_N},$$  

$$\langle P_z \rangle' = \frac{q(E_z)}{v_z} + \frac{dP_z}{ds} - mc \epsilon_N (\beta_\perp \kappa - \mathcal{L}) \frac{qB'}{P_z}.$$  

This set of equations allows for interactions with material in addition to arbitrary changes in beam momentum. The equation for emittance growth is not new [7], but the coupled transport and cooling equations first appeared in [8]. A computer code for the evolution of moments based on symbolic manipulation of the Vlasov equation has been developed by Shadwick [9]. The formalism in this paper differs mainly by the development of an extension of the conventional beam dynamics parameters, which facilitates analytic study. In addition, analytic expressions for the beam cooling rates in terms of material properties are given in Ref. [10], for a fixed momentum beam in a lattice with a given beta function.

In a vacuum with only magnetic fields, the beam parameters evolve according to $\beta_\perp = -2\alpha_\perp$ and

$$2\beta_\perp \beta_\perp' - (\beta_\perp')^2 + 4\beta_\perp^2 \kappa^2 - 4(1 + \mathcal{L}^2) = 0.$$  

$\mathcal{L}$ and $\epsilon_N$ are constant, and $\gamma_\perp$ is given by Eq. (5). Note that $\beta_\perp = \beta_\perp \sqrt{1 + \mathcal{L}^2}$ relates the envelope beta function to the single particle $\beta_\perp$. Thus, canonical angular momentum makes beams harder to focus. In previous treatments [11], the envelope equation includes this contribution to the beam spot size by defining an “effective” emittance related to angular momentum; however, this is not useful in the context of ionization cooling, where only the uncorrelated spread of angles (i.e., true emittance) reduces the effect of multiple scattering.

This theory enables a straightforward analysis of a wide range of cooling channel geometries. It agrees well with a thin lens approximation, which is analytically tractable but not realistic for proposed muon cooling channels. Here, we study a more realistic model with extended solenoids. The field on axis is expanded in three Fourier harmonics in the form $B(z) = B_1 \sin(2\pi z/L) + B_2 \sin(4\pi z/L) + B_3 \sin(6\pi z/L)$, where $L$ is the periodicity of the magnetic field. For a given momentum $P_z$ and charge $|q| = e$, the lattice is characterized by the relative sizes of $B_1$, $B_2$, and $B_3$, together with the quantity

$$\chi = \frac{B_{\text{max}}[T]}{P_z \text{[GeV/c]}^2} \approx 6.67 \kappa_{\text{max}} L.$$  

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where \( B_{\text{max}} \) is the maximum amplitude of the magnetic field on axis. The transport properties of channels without acceleration or material can be described by the momentum acceptance \( \Delta P/P \), defined as the minimum relative momentum shift which will put the beam into unstable transverse motion, and \( \beta_{\perp}/L \).

We first consider the so-called FOFO (“focusing-focusing”) lattice where the magnetic field on axis is a simple sinusoid \( (B_2 = B_3 = 0) \). The momentum acceptance of a FOFO channel has been examined in detail by Fernow [12], starting from the equation for the beam size, \( R \propto \sqrt{\beta} \), which in a vacuum with no electric fields is determined by \( R'' + R - m^2 c^2 e \Phi(1 + L^2)/R^3 P^2 z = 0 \).

The unstable regions were found by neglecting the \( 1/R^3 \) term, and looking for unbounded solutions of the resulting Mathieu equation. These results agree with numerical calculations of periodic solutions to Eq. (8). For the FOFO lattice, all beams with \( \chi < 48.0 \) can be propagated by the channel; this corresponds to all momenta above some resonant value, at which the phase advance per half period is 180 degrees. Immediately below this momentum, particles undergo unstable motion in the transverse direction; however, there are other momentum intervals which support stable motion.

In addition to determining the momentum acceptance, the envelope equations give detailed information about the beta function. At low \( \chi \), \( \beta_{\perp}(z) \) is roughly constant at \( \approx 9.4L/\chi \); the focusing is determined by the average of \( B_2(z) \). The minimum and maximum of the beta function, which, respectively, determine the cooling potential of and the aperture required for a given beam, are shown in Fig. 1 for \( \chi < 48.0 \); a rough numerical fit is

\[
\beta_{\text{min/\max}} = L \frac{9.4}{\chi} \left[ 1 - \left( \frac{\chi}{48.0} \right)^2 \right]^{\pm 1/2} \tag{10}
\]

Close to resonance, this implies that the minimum beta scales as \( \beta_{\text{min}} \approx 0.28L(\Delta P/P)^{1/2} \), where \( \Delta P/P = 1 - \left( \chi/48.0 \right) \). Thus, although the minimum of the beta function can be reduced by taking the beam momentum closer to resonance, the cost in terms of the reduction to the momentum acceptance is high. The maximum of the beta function reaches its smallest value of \( \approx 0.32L \), at \( \chi \approx 34 \). The technological difficulty in achieving a short periodicity \( L \) limits the cooling performance of FOFO channels.

In Figure 2, “phase diagrams” delineating the boundaries between bounded and unbounded motion are shown for the two sets of cases where \( B_3 = 0 \) or \( B_2 = 0 \). The FOFO geometry corresponds to \( B_2 = B_3 = 0 \). Other previously described cooling channel configurations [13] are analogous to various positions in the second interval of stable motion. Note that for \( B_2 = 0 \), \( B_3/B_1 \approx 0.5 \), the first region of unstable motion becomes extremely small.

In the center of the second region of stability, the minimum of the beta function typically scales as \( (\Delta P/P)^{1/2} \) when the geometry of the lattice is altered. In contrast, when the momentum is shifted closer to resonance, \( \beta_{\text{min}} \approx (\Delta P/P)^{1/2} \), as in the case for the FOFO lattice. Thus, it is most efficient to center the region of momentum accep-
tance about the target momentum, and to achieve small $\beta_{\text{min}}$ by varying the geometry.

Solutions to the moment equations yield good agreement with single particle tracking codes, such as ICOOL [14]. As an example, consider a FOFO cooling channel using liquid hydrogen vessels that has been proposed for a neutrino factory. The magnetic field period is 2.2 m, the peak magnetic field on axis is 3.4 T, and the beam momentum is 0.2 GeV/c. The geometry is indicated in Fig. 3 and the transverse emittances from simulations are shown in Fig. 4. There is close agreement between the moment equations and simulation results, although in the simulation which incorporates realistic apertures for the beam line there is an initial sharp drop in emittance due to beam scraping.

To illustrate the effect of canonical angular momentum, we consider a 0.2 GeV/c beam propagating in a uniform 5 T solenoid. Liquid hydrogen vessels and rf are arranged in a 1.1 m period lattice. After 88 meters, there is a sharp field reversal into a ~5 T solenoid. The resultant transverse emittance and canonical momentum are shown in Fig. 5. The results of an ICOOL particle simulation are also shown. Without the field flip to reverse the buildup of angular momentum, the beam emittance would saturate at a significantly higher value.

A paraxial theory for beam moment equations in solenoid fields has been developed, and applied towards lattices designed for ionization cooling of muons. This theory is similar in form to the Courant-Snyder formalism for quadrupole focusing systems, and allows for a rapid analysis of cooling channel performance, including the development of scaling laws. These results are consistent with particle tracking codes and require a small fraction of the computational cost. Extensions of this theory to the full six-dimensional phase space will allow treatment of beam asymmetries, bending magnets, error analysis, nonlinear fields, and space-charge effects.

We wish to thank our colleagues in the Muon Collider and Neutrino Factory Collaboration for many useful and lively discussions. This work was supported by the U.S. Department of Energy Division of High Energy Physics.

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